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The unified Kadomtsev–Petviashvili equation for interfacial waves

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In this paper, the propagation of interfacial waves in a two-layered fluid system is investigated. The interfacial waves are weakly nonlinear and dispersive and propagate in a slowly rotating channel with varying topography and sidewalls, and a weak steady background current field. An evolution equation for the interfacial displacement is derived for waves propagating predominantly in the longitudinal direction of the channel. This new evolution equation is called the unified Kadomtsev–Petviashvili (uKP) equation because most of the KP-type equations existing in the literature for both surface water waves and interfacial waves are special cases of the new evolution equation. The Painlevé PDE test is used to find the conditions under which the uKP equation can be solved by the inverse scattering transform. When these conditions are satisfied, elementary transformations are found to reduce the uKP equation to one of the completely integrable equations: the KP, the Korteweg–de Vries (KdV) or the cylindrical KdV equations. The integral invariants associated with the uKP equation for waves propagating in a varying channel are obtained and their relations with the conservation of mass and energy are discussed.

1. Introduction

The Kadomtsev–Petviashvili (KP) equation was first derived to describe weakly nonlinear and weakly dispersive surface water waves propagating over a constant depth in a predominant direction with a small transversal modulation (Kadomtsev & Petviashvili 1970; Johnson 1980). The constant-depth assumption puts a great limitation on the practical application of the KP equation. Several extended or generalized KP equations have been derived to include additional physical and geometrical factors, such as the Coriolis force, a weak steady background current field with non-vanishing vorticity, and the variation of topography and sidewalls. Moreover, many attempts have also been made to extend the KP equation to internal waves in stratified fluids and interfacial waves in a two-layered system.

Using the Lagrangian equations, Grimshaw (1985) derived a rotation-modified KP (rmKP) equation for long internal waves propagating in a rotating channel with a constant depth and width. Katsis & Akylas (1987) gave an informal derivation of the rotation-modified KP equation for interfacial waves of a two-layered system. They studied the effect of rotation on the propagation of an initially straight-crested Kelvin solitary wave in a rotating channel which has a constant depth and width. In the case of free surface waves, Grimshaw & Melville (1989) rederived the rotation-modified

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KP equation from the Euler equations. They showed that, in general, solutions of the rmKP equation are not locally confined because of the radiation of three-dimensional Poincaré waves behind. Later, Grimshaw & Tang (1990) studied the rmKP equation both analytically and numerically to determine the structure of the solutions which are initially localized.

Djordjevic & Redekopp (1978) gave the variable-coefficient KP (vcKP) equation for interfacial waves propagating over a bottom allowed to vary slowly in the primary wave propagation direction. Starting from the Euler equations, David, Levi & Winternitz (1987) derived a generalized KP (gKP) equation which describes surface water wave propagation in a wide strait or channel with a slowly varying topography and width, and a weak steady current field with non-vanishing vorticity. Under certain restrictions on the vorticity and the geometry of the strait, the gKP equation can be reduced to one of several completely integrable partial differential equations, such as the KP, Korteweg–de Vries (KdV) and cylindrical KdV (cKdV) equations (David, Levi & Winternitz 1989). Iizuka & Wadati (1992) used the potential theory to derive a variable-coefficient KP equation for surface water waves propagating over an uneven bottom in an unbounded domain. Imposing some limitations on the topography, they reduced the vcKP equation into the KP equation and found analytical solutions to describe the deformation of a line soliton due to the depth variation.

In summary, for surface water waves, the existing KP-type equations take either the effect of topographical variation (David *et al.* 1987; Iizuka & Wadati 1992) or the rotation effect into consideration (Grimshaw & Melville 1989), but none of them considers both effects simultaneously. For interfacial waves in a two-layered system with the presence of rotation, the KP-type equation has not been rigorously derived yet. The primary objective of this paper is to derive a unified KP equation (uKP) for surface and interfacial waves propagating in a rotating channel or strait with varying topography and sidewalls. We shall demonstrate that the uKP equation includes most of the existing KP-type equations in the literature as special cases and shall also investigate the properties of the uKP equation.

In the next section (§ 2), we start with the Euler equations for interfacial waves in a two-layered rotating channel. Assuming that the nonlinearity, dispersion, rotation, transversal modulation, and the variation of the topography and the sidewalls of the channel are small and equally important, we derive an evolution equation for the interfacial displacement (details are given in the Appendix), called the unified KP (uKP) equation (see (2.14)). The effect of a weak steady current field on wave propagation is also taken into account in the process of deriving the uKP equation. When the density of the upper layer is zero, the uKP equation reduces to the evolution equation for free surface waves propagating in the same physical and geometrical setting. Most of the existing KP-type equations for surface and interfacial waves are shown to be special cases of the uKP equation. In § 3, the Painlevé PDE test is used to find the complete integrability conditions for the uKP equation, which allow the corresponding Cauchy problem to be solved exactly by the inverse scattering transform. Moreover, when the integrability conditions are satisfied, the uKP equation can be transformed into one of well-known equations: the KP, the KdV or the cKdV equations, via elementary transformations. As a result, for certain topographies and sidewalls, it is possible to obtain analytical solutions for solitary-wave propagation in the absence of rotation (which is one of the conditions for the uKP to be completely integrable according to the Painlevé test). In § 4, the integral invariants associated with the uKP equation for waves propagating in a varying channel are sought and their relations with the conservation of mass and energy are also discussed.

2. The unified KP equation

2.1. *Governing equations and assumptions*

We consider internal waves propagating along the interface of two fluid layers confined to a channel rotating on the f-plane with a constant Coriolis parameter f. The densities of the upper and lower layers are $\tilde{\rho}^+$ and $\tilde{\rho}^-$ ($\tilde{\rho}^- > \tilde{\rho}^+$), respectively. Cartesian coordinates are employed with the $\tilde{z}$-axis pointing vertically upwards, the $\tilde{x}$-axis pointing in the longitudinal direction of the channel and the $\tilde{y}$-axis in the transversal direction. The still interfacial surface is defined by $\tilde{z} = 0$ and the upper and lower layers are originally bounded by $\tilde{z} = \tilde{H}^+$ and $\tilde{z} = -\tilde{H}^-(\tilde{x}, \tilde{y})$ respectively, where the bottom is allowed to vary in the $\tilde{x}$- and $\tilde{y}$-directions.

The fluids in the channel are assumed to be inviscid and incompressible. The governing equations for flows in the upper and lower layers are

$$\frac{\partial \tilde{u}^\pm}{\partial \tilde{x}} + \frac{\partial \tilde{v}^\pm}{\partial \tilde{y}} + \frac{\partial \tilde{w}^\pm}{\partial \tilde{z}} = 0, \tag{2.1a}$$

$$\frac{\partial \tilde{u}^\pm}{\partial \tilde{t}} + \tilde{u}^\pm \frac{\partial \tilde{u}^\pm}{\partial \tilde{x}} + \tilde{v}^\pm \frac{\partial \tilde{u}^\pm}{\partial \tilde{y}} + \tilde{w}^\pm \frac{\partial \tilde{u}^\pm}{\partial \tilde{z}} - f\tilde{v}^\pm = -\frac{1}{\tilde{\rho}^\pm}\frac{\partial \tilde{p}^\pm}{\partial \tilde{x}}, \tag{2.1b}$$

$$\frac{\partial \tilde{v}^\pm}{\partial \tilde{t}} + \tilde{u}^\pm \frac{\partial \tilde{v}^\pm}{\partial \tilde{x}} + \tilde{v}^\pm \frac{\partial \tilde{v}^\pm}{\partial \tilde{y}} + \tilde{w}^\pm \frac{\partial \tilde{v}^\pm}{\partial \tilde{z}} + f\tilde{u}^\pm = -\frac{1}{\tilde{\rho}^\pm}\frac{\partial \tilde{p}^\pm}{\partial \tilde{y}}, \tag{2.1c}$$

$$\frac{\partial \tilde{w}^\pm}{\partial \tilde{t}} + \tilde{u}^\pm \frac{\partial \tilde{w}^\pm}{\partial \tilde{x}} + \tilde{v}^\pm \frac{\partial \tilde{w}^\pm}{\partial \tilde{y}} + \tilde{w}^\pm \frac{\partial \tilde{w}^\pm}{\partial \tilde{z}} = -\frac{1}{\tilde{\rho}^\pm}\frac{\partial \tilde{p}^\pm}{\partial \tilde{z}}, \tag{2.1d}$$

where signs are vertically ordered and superscripts + and − are used to identify quantities in the upper and lower layers, respectively; $\tilde{u}^\pm, \tilde{v}^\pm$ and $\tilde{w}^\pm$ represent the velocity components and $\tilde{p}^\pm$ is the hydrodynamic pressure. The total pressure $\tilde{P}^\pm$ is written as

$$\tilde{P}^\pm = -\tilde{\rho}^\pm g\tilde{z} + \tilde{p}^\pm, \tag{2.1e}$$

where g is the gravitational acceleration.

The kinematic and dynamic boundary conditions along the interface, $\tilde{z} = \tilde{\eta}(\tilde{t}, \tilde{x}, \tilde{y})$, are

$$\tilde{w}^\pm = \frac{\partial \tilde{\eta}}{\partial \tilde{t}} + \tilde{u}^\pm \frac{\partial \tilde{\eta}}{\partial \tilde{x}} + \tilde{v}^\pm \frac{\partial \tilde{\eta}}{\partial \tilde{y}} \quad \text{on} \quad \tilde{z} = \tilde{\eta}, \tag{2.1f}$$

$$\tilde{P}^+ = \tilde{P}^- \quad \text{on} \quad \tilde{z} = \tilde{\eta}. \tag{2.1g}$$

The rigid-lid assumption is adopted to approximate the free surface

$$\tilde{w}^+ = 0 \quad \text{on} \quad \tilde{z} = \tilde{H}^+. \tag{2.1h}$$

The no-flux boundary conditions on the bottom, $\tilde{z} = -\tilde{H}^-(\tilde{x}, \tilde{y})$, and the vertical sidewalls of the channel, $\tilde{y} = \tilde{y}_r(\tilde{x})$ and $\tilde{y} = \tilde{y}_l(\tilde{x})$, are

$$\tilde{w}^- = -\tilde{u}^- \frac{\partial \tilde{H}^-}{\partial \tilde{x}} - \tilde{v}^- \frac{\partial \tilde{H}^-}{\partial \tilde{y}} \quad \text{on} \quad \tilde{z} = -\tilde{H}^-(\tilde{x}, \tilde{y}) \tag{2.1i}$$

and

$$\tilde{v}^{\pm} = \tilde{u}^{\pm}\frac{\mathrm{d}\tilde{y}}{\mathrm{d}\tilde{x}} \quad \text{on} \quad \tilde{y} = \tilde{y}_r(\tilde{x}),\ \tilde{y}_l(\tilde{x}), \tag{2.1j}$$

respectively.

We introduce the following dimensionless variables:

$$\left.\begin{aligned} &(\tilde{x},\tilde{y}) = l_0(x,y), \quad \tilde{z} = h_0 z, \quad \tilde{t} = \frac{l_0}{c_0}t, \quad \tilde{H}^{\pm} = h_0 H^{\pm}, \quad \tilde{y}_l = l_0 y_l, \\ &\tilde{y}_r = l_0 y_r, \quad \tilde{\eta} = a_0\eta, \quad (\tilde{u}^{\pm},\tilde{v}^{\pm}) = \frac{a_0 c_0}{h_0}(u^{\pm},v^{\pm}), \quad \tilde{w}^{\pm} = \frac{a_0 c_0}{l_0}w^{\pm}, \\ &\tilde{\rho}^{\pm} = \rho_0\rho^{\pm}, \quad c_0^2 = (\rho^- - \rho^+)gh_0, \quad \tilde{p}^{\pm} = \frac{a_0}{h_0}c_0^2\rho_0\rho^{\pm}p^{\pm}, \end{aligned}\right\} \tag{2.2}$$

where l_0 and h_0 are the characteristic wavelength and depth, respectively; ρ_0 is the characteristic density; a_0 and c_0 are the characteristic amplitude and phase velocity of linear long interfacial waves, respectively.

The corresponding dimensionless version of equations and boundary conditions (2.1) becomes

$$\frac{\partial u^{\pm}}{\partial x} + \frac{\partial v^{\pm}}{\partial y} + \frac{\partial w^{\pm}}{\partial z} = 0, \tag{2.3a}$$

$$\frac{\partial u^{\pm}}{\partial t} + \epsilon\left(u^{\pm}\frac{\partial u^{\pm}}{\partial x} + v^{\pm}\frac{\partial u^{\pm}}{\partial y} + w^{\pm}\frac{\partial u^{\pm}}{\partial z}\right) - \gamma v^{\pm} = -\frac{\partial p^{\pm}}{\partial x}, \tag{2.3b}$$

$$\frac{\partial v^{\pm}}{\partial t} + \epsilon\left(u^{\pm}\frac{\partial v^{\pm}}{\partial x} + v^{\pm}\frac{\partial v^{\pm}}{\partial y} + w^{\pm}\frac{\partial v^{\pm}}{\partial z}\right) + \gamma u^{\pm} = -\frac{\partial p^{\pm}}{\partial y}, \tag{2.3c}$$

$$\mu^2\left[\frac{\partial w^{\pm}}{\partial t} + \epsilon\left(u^{\pm}\frac{\partial w^{\pm}}{\partial x} + v^{\pm}\frac{\partial w^{\pm}}{\partial y} + w^{\pm}\frac{\partial w^{\pm}}{\partial z}\right)\right] = -\frac{\partial p^{\pm}}{\partial z}, \tag{2.3d}$$

$$w^{\pm} = \frac{\partial \eta}{\partial t} + \epsilon\left(u^{\pm}\frac{\partial \eta}{\partial x} + v^{\pm}\frac{\partial \eta}{\partial y}\right) \quad \text{on} \quad z = \epsilon\eta, \tag{2.3e}$$

$$\rho^+p^+ - \rho^-p^- + \eta = 0 \quad \text{on} \quad z = \epsilon\eta, \tag{2.3f}$$

$$w^+ = 0 \quad \text{on} \quad z = H^+, \tag{2.3g}$$

$$w^- = -u^-\frac{\partial H^-}{\partial x} - v^-\frac{\partial H^-}{\partial y} \quad \text{on} \quad z = -H^-(x,y), \tag{2.3h}$$

$$v^{\pm} = u^{\pm}\frac{\mathrm{d}y}{\mathrm{d}x} \quad \text{on} \quad y = y_r(x), y_l(x), \tag{2.3i}$$

where parameters ϵ, μ^2 and γ are defined as

$$\epsilon = a_0/h_0, \qquad \mu^2 = (h_0/l_0)^2, \qquad \gamma = l_0 f/c_0. \tag{2.4}$$

Thus, ϵ measures the nonlinearity, whereas μ^2 represents the relative shallowness of the fluid layers. The parameter γ is the reciprocal of the Rossby number and measures the ratio of the Coriolis acceleration to the inertial acceleration.

We shall derive an evolution equation for weakly nonlinear and weakly dispersive waves in a slowly rotating channel. Explicitly, we assume that

$$\mu^2 = \alpha\epsilon, \qquad \gamma = \beta\epsilon^{1/2}, \qquad \text{with } \epsilon \ll 1, \tag{2.5}$$

where $\alpha = O(1)$, $\beta = O(1)$ are two arbitrary constants. Furthermore, the weakly three-dimensional effect is also considered (i.e. waves propagate predominantly in one direction, say the $+x$-direction). According to the linear dispersion relation, if the weakly three-dimensional effect is as important as the weakly dispersive effect, the variation of the wave field in the y-direction should be $O(\mu)$ or $O(\epsilon^{1/2})$ by (2.5) (Akylas 1994). In other words, the wave field in the y-direction is a function of a slow variable Y defined as

$$Y = \epsilon^{1/2} y \qquad \text{with} \qquad \frac{\partial}{\partial Y} = \epsilon^{-1/2}\frac{\partial}{\partial y} = O(1). \tag{2.6}$$

Therefore, the channel can be viewed as a very wide channel in the sense that its typical width $l_0/\epsilon^{1/2} = O(h_0/\epsilon)$ is much greater than the typical depth h_0 for $\epsilon \ll 1$.

If the effects of the variation of the topography and the sidewalls of the channel are as important as the effects of nonlinearity, dispersion, rotation and transversal modulation, the assumptions (2.5) and (2.6) impose certain limitations on the shapes of the topography and the sidewalls of the channel. The most general topography and sidewalls fitting in this framework are

$$H^- = h^-(\epsilon x) + \epsilon B(\epsilon x, Y), \tag{2.7}$$

$$y_r = \epsilon^{-1/2} Y_R(\epsilon x), \qquad y_l = \epsilon^{-1/2} Y_L(\epsilon x). \tag{2.8a, b}$$

In other words, the topographical variation in the y-direction is both gentle (H^- is a function of slow variable Y) and weak ($\partial_Y H^- = O(\epsilon)$), while the variation in the x-direction is only required to be gentle. The channel should be wide and the change of the sidewalls in the x-direction should be gentle. We remark here that in David *et al.*'s (1987) paper for surface water waves, the topography in the y-direction is restricted to a linear function of Y (up to $O(\epsilon)$): $h = h_0(\epsilon x) + \epsilon h_1(\epsilon x) Y + O(\epsilon^2)$, which is a special case of (2.7).

2.2. *Perturbation analysis*

We introduce the following transformation:

$$\xi = \int_0^x C^{-1}(\epsilon x)\mathrm{d}x - t, \qquad X = \epsilon x, \qquad Y = \epsilon^{1/2} y, \qquad Z = z, \tag{2.9a}$$

where

$$C(X) = \left(\rho^+/h^+ + \rho^-/h^-\right)^{-1/2} \tag{2.9b}$$

and $h^+ \equiv H^+ \equiv$ constant is used. Note that h^- is the leading-order term in the expression for the topography H^- (see (2.7)). Therefore, C is the leading order of the local linear-long-wave speed and $\xi = O(1)$ is the characteristic coordinate moving at the speed of C. Transformation (2.9*a*) is simpler than the equivalent one used by David *et al.* (1987) for surface water waves (see (2.17) and (2.21) in their paper), because by definition (2.9*b*), $C(X)$ is independent of ϵ and Y in our new coordinates. Thus, there is no need to expand C in terms of ϵ in the following perturbation analysis and the relations among the derivatives in the new and old coordinates are much simpler.

The relations between the derivatives with respect to the old independent variables (t, x, y, z) and the new independent variables (ξ, X, Y, Z) are given by

$$\frac{\partial}{\partial t} = -\frac{\partial}{\partial \xi}, \quad \frac{\partial}{\partial x} = \frac{1}{C}\frac{\partial}{\partial \xi} + \epsilon\frac{\partial}{\partial X}, \quad \frac{\partial}{\partial y} = \epsilon^{1/2}\frac{\partial}{\partial Y}, \quad \frac{\partial}{\partial z} = \frac{\partial}{\partial Z}. \tag{2.10}$$

In terms of the new coordinates, equations and boundary conditions (2.3) can be rewritten as

$$\frac{1}{C}\frac{\partial u^{\pm}}{\partial \xi}+\epsilon\frac{\partial u^{\pm}}{\partial X}+\epsilon^{1/2}\frac{\partial v^{\pm}}{\partial Y}+\frac{\partial w^{\pm}}{\partial Z}=0, \tag{2.11a}$$

$$\frac{\partial u^{\pm}}{\partial \xi}-\epsilon\left[u^{\pm}\left(\frac{1}{C}\frac{\partial u^{\pm}}{\partial \xi}+\epsilon\frac{\partial u^{\pm}}{\partial X}\right)+\epsilon^{1/2}v^{\pm}\frac{\partial u^{\pm}}{\partial Y}+w^{\pm}\frac{\partial u^{\pm}}{\partial Z}\right]+\epsilon^{1/2}\beta v^{\pm}$$
$$=\frac{1}{C}\frac{\partial p^{\pm}}{\partial \xi}+\epsilon\frac{\partial p^{\pm}}{\partial X}, \tag{2.11b}$$

$$\frac{\partial v^{\pm}}{\partial \xi}-\epsilon\left[u^{\pm}\left(\frac{1}{C}\frac{\partial v^{\pm}}{\partial \xi}+\epsilon\frac{\partial v^{\pm}}{\partial X}\right)+\epsilon^{1/2}v^{\pm}\frac{\partial v^{\pm}}{\partial Y}+w^{\pm}\frac{\partial v^{\pm}}{\partial Z}\right]-\epsilon^{1/2}\beta u^{\pm}=\epsilon^{1/2}\frac{\partial p^{\pm}}{\partial Y}, \tag{2.11c}$$

$$\epsilon\alpha\left\{\frac{\partial w^{\pm}}{\partial \xi}-\epsilon\left[u^{\pm}\left(\frac{1}{C}\frac{\partial w^{\pm}}{\partial \xi}+\epsilon\frac{\partial w^{\pm}}{\partial X}\right)+\epsilon^{1/2}v^{\pm}\frac{\partial w^{\pm}}{\partial Y}+w^{\pm}\frac{\partial w^{\pm}}{\partial Z}\right]\right\}=\frac{\partial p^{\pm}}{\partial Z}, \tag{2.11d}$$

$$w^{\pm}=-\frac{\partial \eta}{\partial \xi}+\epsilon\left[u^{\pm}\left(\frac{1}{C}\frac{\partial \eta}{\partial \xi}+\epsilon\frac{\partial \eta}{\partial X}\right)+\epsilon^{1/2}v^{\pm}\frac{\partial \eta}{\partial Y}\right] \quad \text{on} \quad Z=\epsilon\eta, \tag{2.11e}$$

$$\rho^{+}p^{+}-\rho^{-}p^{-}+\eta=0 \quad \text{on} \quad Z=\epsilon\eta, \tag{2.11f}$$

$$w^{+}=0 \quad \text{on} \quad Z=h^{+}, \tag{2.11g}$$

$$w^{-}=-\epsilon u^{-}\left(\frac{\mathrm{d}h^{-}}{\mathrm{d}X}+\epsilon\frac{\partial B}{\partial X}\right)-\epsilon^{3/2}v^{-}\frac{\partial B}{\partial Y} \quad \text{on} \quad Z=-h^{-}-\epsilon B, \tag{2.11h}$$

$$v^{\pm}=\epsilon^{1/2}u^{\pm}\frac{\mathrm{d}Y}{\mathrm{d}X} \quad \text{on} \quad Y=Y_R(X),\ Y_L(X), \tag{2.11i}$$

where (2.7) and (2.8) have been used.

A solution to the governing equations and boundary conditions (2.11) is sought in the following series forms:

$$G(\xi,X,Y,Z;\epsilon)=G_0(\xi,X,Y,Z)+\epsilon G_1(\xi,X,Y,Z)+O(\epsilon^2), \tag{2.12a}$$

$$v^{\pm}(\xi,X,Y,Z;\epsilon)=\epsilon^{1/2}\left[v_0^{\pm}(\xi,X,Y,Z)+\epsilon v_1^{\pm}(\xi,X,Y,Z)+O(\epsilon^2)\right], \tag{2.12b}$$

$$\eta(\xi,X,Y;\epsilon)=\eta_0(\xi,X,Y)+\epsilon\eta_1(\xi,X,Y)+O(\epsilon^2), \tag{2.12c}$$

where $G=\{u^{\pm},w^{\pm},p^{\pm}\}$. We remark here that the leading order of the x-component of the velocity, $u_0^{\pm}$, is $O(1)$, whereas the leading order of the y-component, $\epsilon^{1/2}v_0^{\pm}$, is $O(\epsilon^{1/2})$. Substituting (2.12) into (2.11) and expanding the interfacial conditions (2.11*e*) and (2.11*f*) at $Z=0$ and the bottom boundary condition (2.11*h*) at $Z=-h^{-}$, we obtain a sequence of initial-boundary-value problems by collecting coefficients of ϵ^n. In the zeroth-order problem ($n=0$), a steady background current field, which appears as constants of integration with respect to ξ, is taken into consideration. All the zeroth-order variables are expressed in terms of η_0 which can be determined from the solvability condition of the first-order problem ($n=1$). The details are given in the Appendix. If we write

$$\eta_0=\bar{\eta}(X,Y)+\hat{\eta}(\xi,X,Y), \tag{2.13}$$

where $\bar{\eta}$ is the mean interfacial displacement and can be predetermined through the rotation and the background current field (see (A 16) and note that when the rotation is absent or the upper- and lower-layer averaged mass fluxes of the background current in the x-direction are equal, $\bar{\eta}$=0), the solvability condition from the first-order problem requires that the unsteady part of the leading-order displacement, $\hat{\eta}$, should satisfy the following evolution equation (see (A 17)):

$$C^{1/2}\frac{\partial}{\partial X}\left(C^{1/2}\frac{\partial \hat{\eta}}{\partial \xi}\right)+\frac{3C^2}{4}D_{-2}\frac{\partial^2 \hat{\eta}^2}{\partial \xi^2}+\frac{\alpha D_1}{6}\frac{\partial^4 \hat{\eta}}{\partial \xi^4}+\frac{C^2}{2}\frac{\partial^2 \hat{\eta}}{\partial Y^2}-\frac{\beta^2}{2}\hat{\eta} + \left[\frac{\rho^- B C^2}{2(h^-)^2}+C\mathscr{N}_2+\frac{3C^2}{2}D_{-2}\bar{\eta}\right]\frac{\partial^2 \hat{\eta}}{\partial \xi^2}=0. \tag{2.14}$$

The coefficients D_1 and D_{-2} and $\mathscr{N}_2$ in (2.14) are defined as

$$D_n(X)=\rho^-(h^-)^n+(-1)^{(n-1)}\rho^+(h^+)^n, \quad (n=1,\ -2) \tag{2.15a}$$

and

$$\mathscr{N}_2(X,Y)=\frac{\rho^-}{(h^-)^2}\int_{-h^-}^{0}\mathscr{F}_u^-\mathrm{d}Z+\frac{\rho^+}{(h^+)^2}\int_0^{h^+}\mathscr{F}_u^+\mathrm{d}Z, \tag{2.15b}$$

where $\mathscr{F}_u^{\pm}(X,Y,Z)$ represent the x-component velocities of the background steady current in the upper- and lower-layer, respectively. The boundary conditions for $\hat{\eta}$ can be written as (see (A 18))

$$\frac{\partial \hat{\eta}}{\partial Y}+\frac{\beta}{C}\hat{\eta}-\frac{1}{C}\frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\partial \hat{\eta}}{\partial \xi}=0 \quad \text{on} \quad Y=Y_R(X),\ Y_L(X). \tag{2.16}$$

In equation (2.14), X, which is proportional to the spatial variable x, can be viewed as a time-like coordinate, whereas ξ as a space-like coordinate. The 'initial' condition for (2.14) at $X=X_0$ corresponds to the interfacial displacement data measured over a period of the physical time t along the cross-section $x=x_0$. The physical meaning of each term in (2.14) is explained as follows: the first term represents refraction and shoaling; if other terms in (2.14) are neglected, then (2.14) yields $\hat{\eta}\propto C^{-1/2}$, which is equivalent to the Green law; the second and the third terms describe the nonlinear and frequency dispersion effects, respectively; the fourth term represents the modulation in the transversal direction; the fifth term accounts for the rotation effect; and the last term comes from the difference between the leading-order linear-long-wave speed C and the actual linear-long-wave speed, which is contributed by the deviation of actual topography from h^-, the background steady current field and the mean interfacial displacement. The effect of the sidewalls appears only in boundary conditions (see (2.16)). In (2.14), the nonlinear term is proportional to $D_{-2}=\rho^-/(h^-)^2-\rho^+/(h^+)^2$. To ensure that the nonlinearity is as important as the dispersion and other effects, we make another assumption that $D_{-2}=O(1)$.

2.3. *Reduction to various KP-type equations*

When the channel is uniform (i.e. the bottom is flat and the sidewalls are straight and parallel), the background current field is absent, and $\alpha=1$ (i.e. $\mu^2=\epsilon$), equation (2.14) and boundary conditions (2.16) become ($\bar{\eta}=0$ and $\hat{\eta}=\eta_0$)

$$C\frac{\partial^2 \eta_0}{\partial X \partial \xi}+\frac{3C^2}{4}D_{-2}\frac{\partial^2 \eta_0^2}{\partial \xi^2}+\frac{D_1}{6}\frac{\partial^4 \eta_0}{\partial \xi^4}+\frac{C^2}{2}\frac{\partial^2 \eta_0}{\partial Y^2}-\frac{\beta^2}{2}\eta_0=0, \tag{2.17a}$$

$$\frac{\partial \eta_0}{\partial Y}+\frac{\beta}{C}\eta_0=0 \quad \text{on} \quad Y=Y_R,\ Y_L. \tag{2.17b}$$

For uniform channels, it is more convenient to use a slow temporal variable $T = \epsilon t$ instead of the slow spatial variable $X = \epsilon x$ in transformation (2.9*a*)

$$\xi = x/C - t, \quad T = \epsilon t, \quad Y = \epsilon^{1/2} y, \quad Z = z. \tag{2.18a}$$

The relation between T and X is given by

$$T = X/C - \epsilon\xi. \tag{2.18b}$$

With the following changes:

$$\frac{\partial}{\partial X} \to \frac{1}{C}\frac{\partial}{\partial T}, \qquad \frac{\partial}{\partial \xi} \to \frac{\partial}{\partial \xi} - \epsilon\frac{\partial}{\partial T}, \tag{2.19}$$

equation (2.17*a*) becomes (after $O(\epsilon)$ terms have been dropped)

$$\frac{\partial^2 \eta_0}{\partial T \partial \xi} + \frac{3C^2}{4} D_{-2} \frac{\partial^2 \eta_0^2}{\partial \xi^2} + \frac{D_1}{6}\frac{\partial^4 \eta_0}{\partial \xi^4} + \frac{C^2}{2}\frac{\partial^2 \eta_0}{\partial Y^2} - \frac{\beta^2}{2}\eta_0 = 0, \tag{2.20}$$

which is the evolution equation, in terms of (T, ξ, Y, Z), for interfacial waves propagating in a uniform channel. The boundary conditions for η_0 in terms of (T, ξ, Y, Z) remain the same as (2.17*b*).

For the rigid-lid assumption on the free surface to be valid, the difference between the densities in the upper and lower layers must be very small. Thus, $\rho^{\pm} \approx 1$ and $D_n \approx (h^-)^n + (-1)^{n-1}(h^+)^n$. To compare equation (2.20) with the rmKP equation given by Katsis & Akylas (1987), we rescale the variables T and ξ such that

$$\frac{\partial}{\partial T} \to C\frac{\partial}{\partial T}, \qquad \frac{\partial}{\partial \xi} \to C\frac{\partial}{\partial \xi} \tag{2.21}$$

and use the depth of the lower layer as the typical depth (i.e. $h^- = 1$ and $h^+ = \tilde{h}^+/\tilde{h}^-$). In so doing, equation (2.20) becomes

$$\frac{\partial^2 \eta_0}{\partial T \partial \xi} + \frac{3}{4}\left(1 - \tilde{h}^-/\tilde{h}^+\right)\frac{\partial^2 \eta_0^2}{\partial \xi^2} + \frac{\tilde{h}^+}{6\tilde{h}^-}\frac{\partial^4 \eta_0}{\partial \xi^4} + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial Y^2} - \frac{\beta^2}{2C^2}\eta_0 = 0. \tag{2.22}$$

The rmKP equation given by the Katsis & Akylas (1987) is for the left-going waves. For the right-going waves, the corresponding rmKP equation can be obtained by changing the signs of all terms except the first term in their rmKP equation ((12) in their paper), which is exactly the same as (2.22) (note that β in their paper is equal to $\frac{1}{2}\beta/C$ in this paper). The boundary conditions given by Katsis & Akylas ((13) in their paper) also agree with (2.17*b*).

For surface water waves, $\rho^+ = 0, \rho^- = 1$ and $h^- = 1$, equation (2.20) and boundary conditions (2.17*b*) become

$$\frac{\partial^2 \eta_0}{\partial T \partial \xi} + \frac{3}{4}\frac{\partial^2 \eta_0^2}{\partial \xi^2} + \frac{1}{6}\frac{\partial^4 \eta_0}{\partial \xi^4} + \frac{1}{2}\frac{\partial^2 \eta_0}{\partial Y^2} - \frac{\beta^2}{2}\eta_0 = 0, \tag{2.23a}$$

$$\frac{\partial \eta_0}{\partial Y} + \beta\eta_0 = 0 \quad \text{on} \quad Y = Y_R,\ Y_L, \tag{2.23b}$$

which agree with the rmKP equation and boundary conditions derived by Grimshaw & Melville (1989). When $\beta = 0$, (2.23*a*) also recovers the KP equation given by Mathew & Akylas (1990) (note that their KP equation (15) is for left-going waves; for right-going waves, all the minus signs in the equation should change into plus signs), whereas the boundary conditions (2.23*b*) for vertical sidewalls reduce to the special case of the sloping sidewall boundary conditions given by Mathew & Akylas (1990) (see (16*a*) with $\mathscr{A}=0$ in their paper).

On the other hand, in the absence of rotation, i.e. β=0, equation (2.14) and boundary conditions (2.16) can be simplified to ($\bar{\eta}=0$ and $\hat{\eta}=\eta_0$)

$$C^{1/2}\frac{\partial}{\partial X}\left(C^{1/2}\frac{\partial\eta_0}{\partial\xi}\right)+\frac{3C^2}{4}D_{-2}\frac{\partial^2\eta_0^2}{\partial\xi^2}+\frac{\alpha D_1}{6}\frac{\partial^4\eta_0}{\partial\xi^4}+\frac{C^2}{2}\frac{\partial^2\eta_0}{\partial Y^2}+\left[\frac{\rho^-BC^2}{2(h^-)^2}+C\mathscr{N}_2\right]\frac{\partial^2\eta_0}{\partial\xi^2}=0, \qquad (2.24a)$$

$$\frac{\partial\eta_0}{\partial Y}-\frac{1}{C}\frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\partial\eta_0}{\partial\xi}=0 \quad \text{on} \quad Y=Y_R(X),\ Y_L(X). \qquad (2.24b)$$

Several further simplifications can be made. We discuss two different situations:

(*a*) If the background current is absent and the topography can be expressed as $H^-=h^-(X)$, i.e. $\mathscr{F}_u^{\pm}=0$ and $B=0$, equation (2.24*a*) is reduced to

$$C^{1/2}\frac{\partial}{\partial X}\left(C^{1/2}\frac{\partial\eta_0}{\partial\xi}\right)+\frac{3C^2}{4}D_{-2}\frac{\partial^2\eta_0^2}{\partial\xi^2}+\frac{\alpha D_1}{6}\frac{\partial^4\eta_0}{\partial\xi^4}+\frac{C^2}{2}\frac{\partial^2\eta_0}{\partial Y^2}=0, \qquad (2.25)$$

which agrees with the vcKP equation for interfacial waves given by Djordjevic & Redekopp (1978) (see (4.16) in their paper).

(*b*) For surface water waves, $\rho^+=0, \rho^-=1$ and $C^2=h^-(X)=h_0(X)$, equation (2.24*a*) and boundary conditions (2.24*b*) become

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\eta_0}{\partial X}+\frac{3}{2}h_0^{-3/2}\eta_0\frac{\partial\eta_0}{\partial\xi}+\frac{\alpha}{6}h_0^{1/2}\frac{\partial^3\eta_0}{\partial\xi^3}\right)+\frac{1}{2}h_0^{1/2}\frac{\partial^2\eta_0}{\partial Y^2}+\frac{\partial}{\partial\xi}\left[\frac{1}{4h_0}\frac{\mathrm{d}h_0}{\mathrm{d}X}\eta_0+\left(\frac{1}{2}h_0^{-3/2}B+h_0^{-2}\int_{-h_0}^{0}\mathscr{F}_u\mathrm{d}Z\right)\frac{\partial\eta_0}{\partial\xi}\right]=0, \qquad (2.26a)$$

$$\frac{\partial\eta_0}{\partial Y}-h_0^{-1/2}\frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\partial\eta_0}{\partial\xi}=0 \quad \text{on} \quad Y=Y_R(X),\ Y_L(X). \qquad (2.26b)$$

If the background current is ignored, $\mathscr{F}_u$=0, equation (2.26*a*) agrees with the vcKP equation given by Iizuka & Wadati (1992). On the other hand, when $\mathscr{F}_u\neq 0$ but $B=0$ (i.e. the topography does not vary in the transversal direction), the transformation used by David *et al.* (1987) ((2.17) with $C=1$ in their paper) is the same as our transformation (2.9*a*). The equation and boundary conditions they derived for this situation agree with (2.26*a*) and (2.26*b*). (Note that $A_i=B_i=0$ and $\phi_0=h_0^{1/2}\mathscr{F}_u$ in (2.31) and (2.33) in their paper.) The variable-coefficient KdV equation derived by Kakutani (1971) and Johnson (1973) can also be recovered by setting $\mathscr{F}_u=B=0, \partial_Y=0$ in equation (2.26*a*).

In view of the discussions given above, we conclude that the evolution equation (2.14) is a general equation and most of the KP-type equations appearing in the literature are special cases of (2.14). Therefore, we call (2.14) the unified KP (uKP) equation for surface and interfacial waves in a rotating channel with varying topography and sidewalls, and a steady background current field.

3. Complete integrability of the uKP equation

The unified KP equation (2.14), together with the initial condition $\hat{\eta}(\xi, X=0, Y)=\eta^0(\xi, Y)$, boundary conditions along the vertical sidewalls (2.16), and appropriate boundary conditions at $\xi = \pm\infty$, describes the evolution of the interfacial elevation $\hat{\eta}$ in a rotating channel with varying topography and sidewalls, and a steady background current field. In this section, our investigation focuses on initial-value problems.

Because of the variable coefficients and the appearance of the rotation term in the uKP equation (2.14), in general, no analytical procedure is available for obtaining solutions of the corresponding Cauchy problem. On the other hand, the KP equation, the KdV equation and the cylindrical KdV equation (cKdV) are completely integrable, i.e. with a suitable initial condition they can be solved by the inverse scattering transform. These equations possess a number of remarkable properties: the existence of soliton solutions, an infinite number of symmetries and conservation laws, similarity reductions to the Painlevé equations, Bäcklund transformations and the Lax representation (Ablowitz & Clarkson 1991).

A powerful tool to investigate the complete integrability of a nonlinear evolution equation is the Painlevé PDE test, which also yields other information such as Lax pairs and Bäcklund transformations (Weiss, Tabor & Carnevale 1983; Clarkson 1990; Brugarino & Greco 1991). The Painlevé PDE test provides a useful criterion for whether a given partial differential equation is completely integrable. It gives the necessary conditions on the coefficients of a nonlinear evolution equation so that all solutions to the evolution equation are 'single-valued' in the neighbourhood of the non-characteristic movable singularity manifold (Ablowitz & Clarkson 1991). Moreover, when these conditions are satisfied, the evolution equation can be reduced to the canonical forms (e.g. KP, KdV or cKdV) via elementary transformations. In the following subsections, we first carry out the Painlevé analysis to search for the conditions under which the uKP equation, (2.14), is completely integrable. When these conditions are satisfied, we seek transformations to reduce the uKP equation to one of the known integrable equations.

3.1. *Painlevé analysis*

To simplify the algebraic manipulation encountered in the Painlevé PDE test, we introduce the following transformation:

$$\hat{\eta} = \frac{4D_1}{C^2 D_{-2}}\zeta, \tag{3.1a}$$

$$\tau = (6/\alpha)^{1/2}\int_0^X D_1/C\,\mathrm{d}X, \quad \theta = (6/\alpha)^{1/2}\xi, \quad \lambda = (6/\alpha)^{1/2}Y. \tag{3.1b}$$

Under this transformation, equation (2.14) becomes

$$\frac{\partial^2\zeta}{\partial\tau\partial\theta} + 3\frac{\partial^2\zeta^2}{\partial\theta^2} + \frac{\partial^4\zeta}{\partial\theta^4} + a(\tau)\frac{\partial^2\zeta}{\partial\lambda^2} + b(\tau,\lambda)\frac{\partial^2\zeta}{\partial\theta^2} + c(\tau)\frac{\partial\zeta}{\partial\theta} - d(\tau)\zeta = 0, \tag{3.2}$$

where

$$a = \frac{C^2}{2D_1}, \tag{3.3a}$$

$$b = \frac{1}{D_1}\left[\frac{\rho^- BC^2}{2(h^-)^2} + C\mathcal{N}_2 + \frac{3C^2}{2}D_{-2}\bar{\eta}\right], \tag{3.3b}$$

$$c = (\alpha/6)^{1/2}\frac{\rho^- C}{D_1(h^-)^3}\left[\frac{2}{D_{-2}} - \frac{3}{4}h^- C^2 + \frac{(h^-)^3}{D_1}\right]\frac{\mathrm{d}h^-}{\mathrm{d}X}, \tag{3.3c}$$

$$d = \frac{\alpha\beta^2}{12D_1}. \tag{3.3d}$$

The Painlevé PDE test for (3.2) consists of seeking conditions on the coefficients a, b, c and d so that the equation admits solutions of the form of a Laurent series

$$\zeta(\tau, \theta, \lambda) = \phi^p \sum_{j=0}^{\infty} u_j(\tau, \lambda)\phi^j, \tag{3.4a}$$

with

$$\phi(\tau, \theta, \lambda) = \theta + \psi(\tau, \lambda), \tag{3.4b}$$

where $\psi(\tau, \lambda)$ and $u_j(\tau, \lambda)$ $(j = 0, 1, 2, \ldots)$ are analytic functions of τ and λ in the neighbourhood of a non-characteristic movable singularity manifold defined by $\phi = 0$ and p is an integer. Substituting (3.4) into (3.2) and equating coefficients of like powers of ϕ, we can determine p and define the recursion relation for $u_j (j = 0, 1, 2, \ldots)$. To pass the Painlevé PDE test, the expansion should be well-defined and contain the maximum number of arbitrary functions allowed (in this case four). The compatibility conditions at each resonance, occurring at some j where u_j is arbitrary, give the conditions for a, b, c and d so that the equation (3.2) will have solutions of the form (3.4).

The analysis of the leading-order term requires $p = -2$ and $u_0 = -2$. Equating the like powers of ϕ yields the general recursion relation

$$(j+1)(j-4)(j-5)(j-6)u_j + W_j = 0, \tag{3.5a}$$

where

$$\begin{aligned} W_j = {} & 3(j-4)(j-5)\sum_{k=1}^{j-1} u_k u_{j-k} + (j-4)(j-5)u_{j-2}\left[\frac{\partial\psi}{\partial\tau} + a\left(\frac{\partial\psi}{\partial\lambda}\right)^2 + b\right] \\ & + (j-5)u_{j-3}\left(c + a\frac{\partial^2\psi}{\partial\lambda^2}\right) + (j-5)\left(\frac{\partial u_{j-3}}{\partial\tau} + 2a\frac{\partial\psi}{\partial\lambda}\frac{\partial u_{j-3}}{\partial\lambda}\right) \\ & + a\frac{\partial^2 u_{j-4}}{\partial\lambda^2} - du_{j-4} \end{aligned} \tag{3.5b}$$

for $j \geqslant 1$ (define u_j=0 for $j < 0$). The recursion relation (3.5) defines u_j for $j \geqslant 1$ unless $j = 4, 5, 6$ where resonances occur (the resonance at $j = -1$ is usually associated with the fact that $\psi(\tau, \lambda)$ is an arbitrary function). Therefore, the recursion relation (3.5) is consistent provided that $W_j \equiv 0$ for $j = 4, 5, 6$, which are the compatibility conditions. From (3.5), we obtain

$$u_1 = 0, \tag{3.6a}$$

$$u_2 = -\frac{1}{6}\left[\frac{\partial\psi}{\partial\tau} + a\left(\frac{\partial\psi}{\partial\lambda}\right)^2 + b\right], \tag{3.6b}$$

$$u_3 = \frac{1}{6}\left(c + a\frac{\partial^2\psi}{\partial\lambda^2}\right). \tag{3.6c}$$

The compatibility condition for $j = 4$ gives

$$d = 0, \qquad \text{i.e.} \qquad \beta = 0, \tag{3.7}$$

whereas the compatibility condition for $j=5$ is automatically satisfied. The compatibility condition for the resonance occurring at $j=6$ yields

$$2c^2+\frac{\mathrm{d}c}{\mathrm{d}\tau}-a\frac{\partial^2 b}{\partial\lambda^2}+\left(4ac+\frac{\mathrm{d}a}{\mathrm{d}\tau}\right)\frac{\partial^2\psi}{\partial\lambda^2}=0. \tag{3.8}$$

Since ψ is an arbitrary function, a, b and c must satisfy the system of equations

$$4ac+\frac{\mathrm{d}a}{\mathrm{d}\tau}=0 \tag{3.9a}$$

and

$$2c^2+\frac{\mathrm{d}c}{\mathrm{d}\tau}-a\frac{\partial^2 b}{\partial\lambda^2}=0. \tag{3.9b}$$

According to the definitions of a, b and c (see (3.3)), in terms of h^-, (3.9a) becomes

$$(\alpha/6)^{1/2}\frac{\rho^- C^3}{(h^-)^3 D_1^2}\left[\frac{8}{D_{-2}}-2h^-C^2+\frac{3(h^-)^3}{D_1}\right]\frac{\mathrm{d}h^-}{\mathrm{d}X}=0, \tag{3.10a}$$

i.e.

$$h^-=\text{const.} \tag{3.10b}$$

Note that D_{-2}, D_1 and C^2 are rational functions of h^- (see (2.15a) and (2.9b)), so setting the factor in the square bracket in (3.10a) equal to zero also yields $h^-=$ constant if the resulting equation has real positive roots. Consequently, from (3.3c), $c=0$ and (3.9b) can be simplified to

$$\frac{\partial^2 b}{\partial\lambda^2}=0, \tag{3.11a}$$

whose solution is

$$b(\tau,\lambda)=f_1(\tau)\lambda+f_0(\tau), \tag{3.11b}$$

where f_0 and f_1 are arbitrary functions.

In summary, from (3.1) and (3.3), in the moving coordinates (ξ, X, Y), the conditions for equations (2.14) to fulfil the Painlevé PDE test are

$$\beta=0, \tag{3.12a}$$

$$h^-=\text{const}, \tag{3.12b}$$

$$\frac{\rho^- BC}{2(h^-)^2}+\mathcal{N}_2=F_1(X)Y+F_0(X), \tag{3.12c}$$

where $F_0(X)$ and $F_1(X)$ are arbitrary functions. In other words, equation (2.14) is completely integrable if the rotation is absent, the bottom is flat up to the leading order, and the difference between the leading-order linear-long-wave speed and the actual linear-long-wave speed is only allowed to be a linear function of Y. Note that in David *et al.*'s (1987) paper, the bottom is expressed as $h=h_0(X)+\epsilon h_1(X)Y+O(\epsilon^2)$. The conditions for the gKP equation derived by them to pass the Painlevé PDE test are: $h_0=1$ and the flux of the background current in the x-direction is a linear function of Y only (Clarkson 1990), which are consistent with conditions (3.12b) and (3.12c).

3.2. *Reduction to the KP equation*

When the conditions (3.12) are satisfied, (2.14) can be simplified to

$$\frac{\partial^2\eta_0}{\partial X\partial\xi}+\frac{3C}{4}D_{-2}\frac{\partial^2\eta_0^2}{\partial\xi^2}+\frac{\alpha D_1}{6C}\frac{\partial^4\eta_0}{\partial\xi^4}+\frac{C}{2}\frac{\partial^2\eta_0}{\partial Y^2}+[F_1(X)Y+F_0(X)]\frac{\partial^2\eta_0}{\partial\xi^2}=0, \tag{3.13}$$

where all the coefficients, except the coefficient of the last term, are constant. We seek a transformation to convert (3.13) into the KP equation.

We find that the following transformation:

$$\eta_0=\frac{4D_1}{C^2D_{-2}}\zeta, \tag{3.14a}$$

$$\bar{T}=(6/\alpha)^{1/2}D_1X/C, \tag{3.14b}$$

$$\bar{X}=(6/\alpha)^{1/2}\left\{\xi-Y\int_0^X F_1(q)\mathrm{d}q-\int_0^X\left[F_0(q)+\frac{C}{2}\left(\int_0^q F_1(s)\mathrm{d}s\right)^2\right]\mathrm{d}q\right\}, \tag{3.14c}$$

$$\bar{Y}=6(D_1/\alpha)^{1/2}\left[Y/C+\int_0^X\int_0^q F_1(s)\mathrm{d}s\mathrm{d}q\right], \tag{3.14d}$$

transforms equation (3.13) into the KP equation

$$\frac{\partial}{\partial\bar{X}}\left(\frac{\partial\zeta}{\partial\bar{T}}+6\zeta\frac{\partial\zeta}{\partial\bar{X}}+\frac{\partial^3\zeta}{\partial\bar{X}^3}\right)+3\frac{\partial^2\zeta}{\partial\bar{Y}^2}=0. \tag{3.15}$$

It is known that the KP equation (3.15) is completely integrable and different kinds of analytical solutions, such as N-line-soliton solutions and periodic solutions, can be obtained (Freeman 1980; Hammack, Scheffner & Segur 1989; Ablowitz & Clarkson 1991). Using transformation (3.14), one can easily obtain the corresponding Lax pair for equation (3.13) from the Lax pair for the KP equation (Clarkson 1990).

For initial-boundary-value problems, in general, the lateral boundary conditions (2.16) will interfere with the integrability of the uKP equation. However, under some circumstances, it is possible that the lateral boundary conditions will not interfere with the integrability of the uKP equation. Under the integrability conditions (3.12) and transformation (3.14), the boundary conditions (2.16) become

$$\frac{\partial\zeta}{\partial\bar{Y}}=(6D_1)^{-1/2}\left[\frac{\mathrm{d}Y_R}{\mathrm{d}X}+C\int_0^X F_1(q)\mathrm{d}q\right]\frac{\partial\zeta}{\partial\bar{X}}\quad\text{on}\quad\bar{Y}=\bar{Y}_R(\bar{T}), \tag{3.16a}$$

$$\frac{\partial\zeta}{\partial\bar{Y}}=(6D_1)^{-1/2}\left[\frac{\mathrm{d}Y_L}{\mathrm{d}X}+C\int_0^X F_1(q)\mathrm{d}q\right]\frac{\partial\zeta}{\partial\bar{X}}\quad\text{on}\quad\bar{Y}=\bar{Y}_L(\bar{T}), \tag{3.16b}$$

where

$$\bar{Y}_R(\bar{T})=6(D_1/\alpha)^{1/2}\left[Y_R/C+\int_0^X\int_0^q F_1(s)\mathrm{d}s\mathrm{d}q\right], \tag{3.16c}$$

$$\bar{Y}_L(\bar{T})=6(D_1/\alpha)^{1/2}\left[Y_L/C+\int_0^X\int_0^q F_1(s)\mathrm{d}s\mathrm{d}q\right]. \tag{3.16d}$$

If the solution to the Cauchy problem of the KP equation has the travelling wave form (in this case the KP equation can be reduced to an ordinary differential equation, which admits solitary wave and cnoidal wave solutions (Chen & Wen 1987))

$$\zeta(\bar{X},\bar{Y},\bar{T})=\zeta(k\bar{X}+l\bar{Y}-\omega\bar{T}), \tag{3.17}$$

where k, l and ω are real constants, and the sidewalls are given by

$$Y_R(X) = \int_0^X \left[(6D_1)^{1/2} l/k - C\int_0^q F_1(s)\mathrm{d}s\right] \mathrm{d}q + Y_R(0), \tag{3.18a}$$

$$Y_L(X) = \int_0^X \left[(6D_1)^{1/2} l/k - C\int_0^q F_1(s)\mathrm{d}s\right] \mathrm{d}q + Y_L(0), \tag{3.18b}$$

then the boundary conditions (3.16) are automatically satisfied (note that the sidewalls given by (3.18) are parallel). In this situation, the boundary conditions (2.16) will not interfere with the integrability of the uKP equation (3.13).

The KP equation (3.15) has a solitary-wave solution (Freeman 1980)

$$\zeta = \tfrac{1}{2}k^2 \mathrm{sech}^2\left[\left(k\bar{X} + l\bar{Y} - \omega\bar{T}\right)/2\right], \qquad \omega = k^3 + 3l^2/k, \tag{3.19}$$

where k and l are constants, which can be determined from the amplitude and the direction of the incident solitary wave. From transformation (3.14), the solitary-wave solution for equation (3.13) is

$$\eta_0 = \frac{2D_1}{C^2 D_{-2}} k^2 \mathrm{sech}^2\left[(6/\alpha)^{1/2}\Phi/2\right], \tag{3.20a}$$

where Φ is the phase function given by

$$\Phi = k\left\{\xi - Y\int_0^X F_1(q)\mathrm{d}q - \int_0^X \left[F_0(q) + \frac{C}{2}\left(\int_0^q F_1(s)\mathrm{d}s\right)^2\right]\mathrm{d}q\right\}$$
$$+\, l(6D_1)^{1/2}\left[Y/C + \int_0^X\int_0^q F_1(s)\mathrm{d}s\mathrm{d}q\right] - \omega D_1 X/C. \tag{3.20b}$$

The crest line of the solitary wave is defined as $\Phi = 0$, i.e.

$$k\xi + \left[l(6D_1)^{1/2}/C - k\int_0^X F_1(q)\mathrm{d}q\right] Y - k\int_0^X \left[F_0(q) + \frac{C}{2}\left(\int_0^q F_1(s)\mathrm{d}s\right)^2\right]\mathrm{d}q$$
$$+\, l(6D_1)^{1/2}\int_0^X\int_0^q F_1(s)\mathrm{d}s\mathrm{d}q - \omega D_1 X/C = 0. \tag{3.21}$$

In the moving coordinates (ξ, X, Y), at different X, the crest line still remains a straight line on the (ξ, Y)-plane. However, its direction will change due to the contribution from F_1, whose relations with the topography and the background current are described by (3.12c). The contribution from F_0 only causes the crest line to translate and changes the speed of the solitary wave in the moving coordinates. In the physical stationary coordinates (t, x, y), the crest line is given (by transformation (2.9)) as

$$\left[l(6D_1)^{1/2}/C - k\int_0^{\epsilon x} F_1(q)\mathrm{d}q\right]\epsilon^{1/2}y + l(6D_1)^{1/2}\int_0^{\epsilon x}\int_0^q F_1(s)\mathrm{d}s\mathrm{d}q - \epsilon x\omega D_1/C$$
$$+\, k\left\{x/C - t - \int_0^{\epsilon x}\left[F_0(q) + \frac{C}{2}\left(\int_0^q F_1(s)\mathrm{d}s\right)^2\right]\mathrm{d}q\right\} = 0. \tag{3.22}$$

Strictly speaking, the crest line at different time t is no longer a straight line on the (x, y)-plane, but the curvature of the crest line is very small.

In the absence of rotation, the background current field has the same effect as the weak variation of the bottom (see (3.12)). Without loss of generality, we ignore the

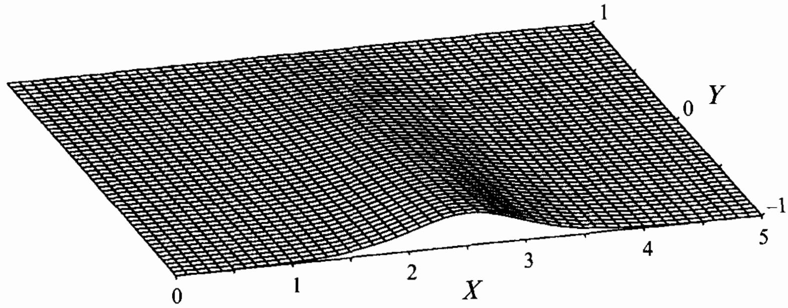

FIGURE 1. The shape of the bottom given by (3.23).

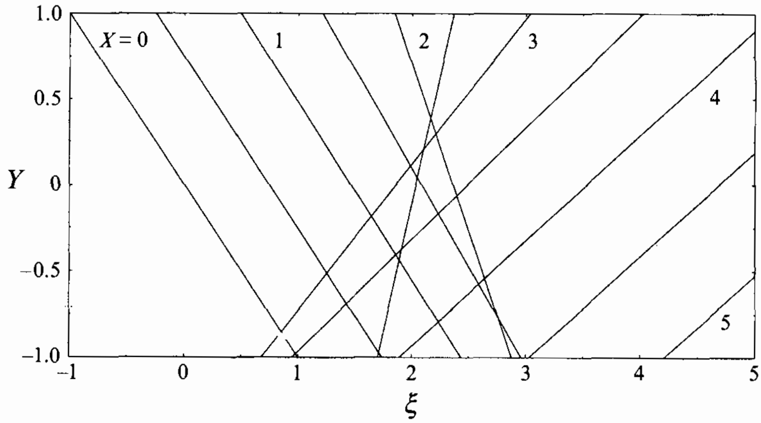

FIGURE 2. The location of the crest line of a solitary wave propagating over the bottom shown in figure 1 at different X in the moving coordinates (X from 0 to 5 with an increment 0.5).

background current field in the following discussions. For simplicity, we only consider a single-layer system, i.e. free surface wave propagation.

For an oblique incident solitary wave initially described by (3.20*a*) with $k = 1$ and $l = 6^{-1/2}$ propagating over a bottom given by

$$H = 1 + \epsilon B = 1 + 2\epsilon (F_1 Y + F_0) = 1 + 4\epsilon \operatorname{sech}^2 [1.5(X - 2.5)]\,(Y - 1), \qquad (3.23)$$

with $\epsilon = 0.05$ (see figure 1), figures 2 and 3 show the location of the crest line in the moving and stationary coordinates, respectively. The lateral sidewalls are not present in this example. In both coordinate systems, the propagation direction of the solitary wave continuously changes from positive angles with respect to the $+\xi$-axis ($+x$-axis) in the moving (stationary) coordinates to negative angles. In the stationary coordinates, this continuous change of directions may be explained by fact that the wave speed of a long wave increases as the depth increases.

Figure 4 shows the location of a normal incident solitary wave propagating in a channel bounded by

$$y_r(x) = -\epsilon^{-1/2} \int_0^{\epsilon x} \int_0^{q} F_1(s) \mathrm{d}s \mathrm{d}q + y_r(0) = -\epsilon^{-1/2} \left[0.003(\epsilon x - 5)(\epsilon x)^4 + 1\right], \quad (3.24a)$$

$$y_l(x) = -\epsilon^{-1/2} \int_0^{\epsilon x} \int_0^{q} F_1(s) \mathrm{d}s \mathrm{d}q + y_l(0) = -\epsilon^{-1/2} \left[0.003(\epsilon x - 5)(\epsilon x)^4 - 1\right], \quad (3.24b)$$

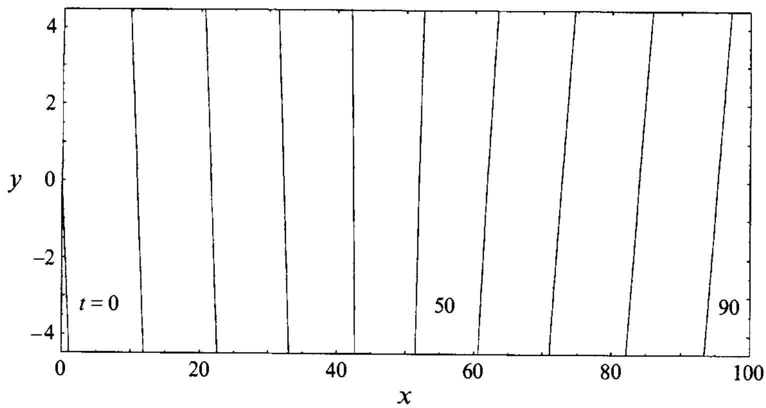

FIGURE 3. The location of the crest line of a solitary wave propagating over the bottom shown in figure 1 at different times t in the stationary coordinates (t from 0 to 90 with an increment 10).

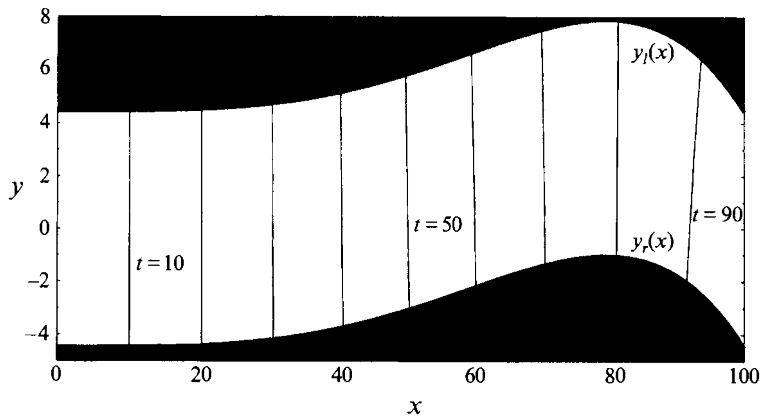

FIGURE 4. The location of the crest line of a normal incident solitary wave propagating over the bottom given by (3.25) in a curved channel as shown in this figure at different times t (t from 0 to 90 with an increment 10).

which are plotted in figure 4 (for $\epsilon = 0.05$). In this case the bottom is given by

$$H = 1 + \epsilon B = 1 + \epsilon \left\{0.12(X^3 - 3X^2)Y - 4\mathrm{sech}^2\left[1.5(X - 2.5)\right]\right\}, \qquad (3.25)$$

which is shown in figure 5.

From (3.22) and (3.24), the primary direction of the crest line coincides with the horizontal slope of the sidewalls at $x \approx t$. If the slope of the sidewalls is positive (negative), then the direction of the crest line is also positive (negative). This agrees with the results shown in figure 4.

3.3. *Reduction to the KdV or cKdV equations*

When the coefficient of the last term in (2.14) is independent of Y, i.e. the topography and the background current do not vary in the transversal direction, a solution which is independent of Y may exist (if the sidewalls are present, they should be straight and parallel). In this situation, the Painlevé PDE test shows that the conditions for

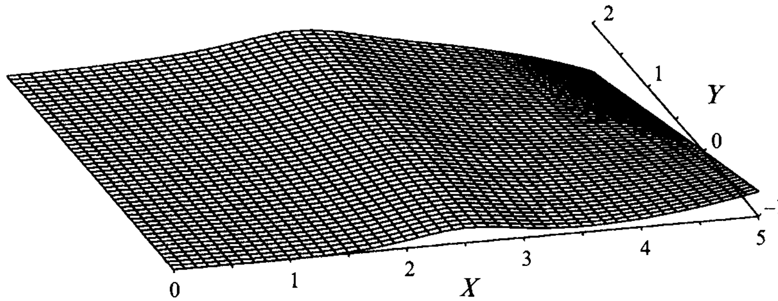

FIGURE 5. The shape of the bottom given by (3.25).

equation (2.14) to be completely integrable are

$$\beta = 0, \tag{3.26a}$$

$$\frac{\mathrm{d}c}{\mathrm{d}\tau} + 2c^2 = 0, \tag{3.26b}$$

where c is given by (3.3c) and

$$\tau = (6/\alpha)^{1/2} \int_0^X D_1/C\mathrm{d}X. \tag{3.26c}$$

The solution to (3.26b) is either $c = 0$ or $c = \frac{1}{2}(\tau + \tau_0)^{-1}$, where τ_0 is a constant.

If conditions (3.26) are satisfied, equation (2.14) can be further simplified to

$$\frac{\partial \eta_0}{\partial X} + \frac{3C}{2} D_{-2}\eta_0 \frac{\partial \eta_0}{\partial \xi} + \frac{\alpha D_1}{6C}\frac{\partial^3 \eta_0}{\partial \xi^3} + F_0(X)\frac{\partial \eta_0}{\partial \xi} + \frac{1}{2C}\frac{\mathrm{d}C}{\mathrm{d}X}\eta_0 = 0, \tag{3.27a}$$

where we have already assumed that $\eta_0 \to 0$ as $\xi \to \pm\infty$ and

$$F_0(X) = \frac{\rho^- BC}{2(h^-)^2} + \mathscr{N}_2. \tag{3.27b}$$

The following transformation:

$$\eta_0 = \frac{4D_1}{C^2 D_{-2}}\zeta, \quad \tau = (6/\alpha)^{1/2}\int_0^X D_1/C\mathrm{d}X, \quad \theta = (6/\alpha)^{1/2}\left[\xi - \int_0^X F_0(q)\mathrm{d}q\right] \tag{3.28}$$

transforms equation (3.27a) into

$$\frac{\partial \zeta}{\partial \tau} + 6\zeta\frac{\partial \zeta}{\partial \theta} + \frac{\partial^3 \zeta}{\partial \theta^3} + c\zeta = 0. \tag{3.29}$$

Since $c = 0$ or $c = \frac{1}{2}(\tau+\tau_0)^{-1}$, (3.29) is either the KdV equation or the cKdV equation. In either case, (3.29) is completely integrable (Gardner *et al.* 1967; Calogero & Degasperis 1978). Moreover, the KdV and cKdV equations are essentially equivalent since their solutions are related by a simple Lie-point transformation (Clarkson 1990). From definition (3.3c) and transformation (3.28), $c = 0$ corresponds to h^-= constant, whereas $c = \frac{1}{2}(\tau + \tau_0)^{-1}$ gives the differential equation for h^-

$$\mathscr{H}\frac{\mathrm{d}^2 h^-}{\mathrm{d}X^2} + \left[\frac{\mathrm{d}\mathscr{H}}{\mathrm{d}h^-} + \frac{\mathscr{H}^2 D_1}{C}\right]\left(\frac{\mathrm{d}h^-}{\mathrm{d}X}\right)^2 = 0, \tag{3.30a}$$

where

$$\mathscr{H}(h^-) = \frac{2\rho^- C}{D_1(h^-)^3}\left[\frac{2}{D_{-2}} - \frac{3}{4}h^- C^2 + \frac{(h^-)^3}{D_1}\right]. \tag{3.30b}$$

For a single-layer system, we can find the analytical solution to (3.30). Substituting $\rho^+ = 0, \rho^- = 1, h^- = h, D_n = h^n$ and $C = h^{1/2}$ into (3.30), we obtain a simple equation for h,

$$h\frac{\mathrm{d}^2 h}{\mathrm{d}X^2} + 3\left(\frac{\mathrm{d}h}{\mathrm{d}X}\right)^2 = 0, \tag{3.31a}$$

whose solution is

$$h(X) = (1 + X/X_0)^{1/4}, \tag{3.31b}$$

where $h(0) = 1$ has been used and $X_0 = \frac{9}{8}(\alpha/6)^{1/2}\tau_0$ can be determined from the slope of the bottom at $X = 0$.

We now give closed-form solutions for one and two solitary waves propagating over an arbitrary weakly and slowly varying topography, which is described by

$$Z = -H^-(X) = -h^- - \epsilon B(X), \tag{3.32}$$

where h^- is a constant. Without loss of generality, we ignore the background current field, since it has the same effect as the topography (3.32) has (see (3.27)).

With the aid of transformation (3.28),

$$\eta_0(\xi, X) = \frac{8D_1}{C^2 D_{-2}}k^2 \mathrm{sech}^2\left[k(6/\alpha)^{1/2}\left(\xi - \frac{\rho^- C}{2(h^-)^2}\int_0^X B\mathrm{d}X - 4k^2 D_1 X/C\right)\right], \tag{3.33}$$

where k is a constant, gives the solution for a solitary wave propagating over a weakly and slowly varying topography (3.32), whereas

$$\eta_0(\xi, X) = \frac{32 D_1}{C^2 D_{-2}} \frac{k_1^2 E_1 + k_2^2 E_2 + 2(k_1 - k_2)^2 E_1 E_2 + A(k_2^2 E_1 + k_1^2 E_2)E_1 E_2}{(1 + E_1 + E_2 + AE_1E_2)^2}, \tag{3.34a}$$

with

$$A = (k_1 - k_2)^2/(k_1 + k_2)^2, \qquad E_i = \exp(2\Theta_i), \qquad i = 1, 2, \tag{3.34b}$$

$$\Theta_i = k_i(6/\alpha)^{1/2}\left[\xi - \frac{\rho^- C}{2(h^-)^2}\int_0^X B\mathrm{d}X - 4k_i^2 D_1 X/C + \delta_i\right], \tag{3.34c}$$

where k_i and δ_i ($i = 1, 2$) are constants, gives the solution for two solitary waves propagating over a weakly and slowly varying topography (3.32) (Drazin & Johnson 1989, pp. 22, 190).

From (3.33) and (3.34), one can see that an individual solitary wave moves with velocity

$$U_i = \frac{\mathrm{d}\xi}{\mathrm{d}X} = \frac{\rho^- BC}{2(h^-)^2} + \frac{4k_i^2 D_1}{C}. \tag{3.35}$$

Depending on the sign of U_i, which is determined by the topography and the amplitude of the solitary wave, the solitary wave will propagate to the right, to the left or remain still. Thus, in the moving coordinates, the effect of the topography (3.32) cannot only change the magnitude of the phase velocity but also change its direction. Therefore, in the moving frame, under different circumstances, a solitary wave can remain still or bounce forward and backward; two solitary waves can collide against each other; a smaller solitary wave can catch up with a larger one after both have reversed their directions. These phenomena cannot exist if the bottom is flat. In the constant-depth case, a solitary wave propagates unidirectionally from the left to the right; a larger solitary wave always propagates faster than a smaller one.

4. Integral invariants

In this section, we seek the integral invariants associated with the uKP equation, (2.14), for waves propagating in a varying channel. Under the assumption that $\hat{\eta}$ is locally confined, i.e. $\hat{\eta}$ and its ξ-derivatives vanish as $\xi \to \pm\infty$, equation (2.14) and boundary conditions (2.16) can be rewritten as

$$\frac{\partial \hat{\eta}}{\partial X} + \frac{1}{2C}\frac{\mathrm{d}C}{\mathrm{d}X}\hat{\eta} + \frac{3C}{4}D_{-2}\frac{\partial \hat{\eta}^2}{\partial \xi} + \frac{\alpha D_1}{6C}\frac{\partial^3 \hat{\eta}}{\partial \xi^3} + \left[\frac{\rho^- BC}{2(h^-)^2} + \mathcal{N}_2 + \frac{3C}{2}D_{-2}\bar{\eta}\right]\frac{\partial \hat{\eta}}{\partial \xi} + \frac{C}{2}\left(\frac{\partial V}{\partial Y} - \frac{\beta}{C}V\right) = 0, \tag{4.1a}$$

$$V(\xi, X, Y) = \int_{+\infty}^{\xi}\left(\frac{\partial \hat{\eta}}{\partial Y} + \frac{\beta}{C}\hat{\eta}\right)\mathrm{d}\xi, \tag{4.1b}$$

$$V = \frac{1}{C}\frac{\mathrm{d}Y}{\mathrm{d}X}\hat{\eta} \quad \text{on} \quad Y = Y_R(X),\ Y_L(X). \tag{4.1c}$$

Letting $\xi \to -\infty$ in (4.1), we obtain

$$V(-\infty, X, Y) = \int_{+\infty}^{-\infty}\left(\frac{\partial \hat{\eta}}{\partial Y} + \frac{\beta}{C}\hat{\eta}\right)\mathrm{d}\xi = 0, \tag{4.2}$$

which implies

$$\int_{-\infty}^{+\infty}\hat{\eta}\,\mathrm{d}\xi = F(X)\exp\left(-\beta Y/C\right), \tag{4.3}$$

where F is an arbitrary function of X. Expression (4.3) shows that if $\hat{\eta}(\xi, X, Y)$ is locally confined, $\int_{-\infty}^{+\infty}\hat{\eta}\mathrm{d}\xi$ along each vertical plane parallel to the ξ-axis must vary exponentially like $\exp\left(-\beta Y/C\right)$ across the channel at different $X > 0$. Expression (4.3) also holds at $X = 0$ if we let $X \to 0^+$. Thus, to hope that the solution of the uKP equation is locally confined (i.e. no wavenumber components with an infinite group velocity are present and thus disturbances remain locally confined), the minimum restriction on the initial condition is that (4.3) is satisfied at $X = 0$ (Grimshaw 1985; Katsis & Akylas 1987; Grimshaw & Melville 1989).

From (4.1), we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}X}\int_{Y_R}^{Y_L}\int_{-\infty}^{+\infty}\hat{\eta}\,\mathrm{d}\xi\mathrm{d}Y + \frac{1}{2C}\frac{\mathrm{d}C}{\mathrm{d}X}\int_{Y_R}^{Y_L}\int_{-\infty}^{+\infty}\hat{\eta}\,\mathrm{d}\xi\mathrm{d}Y \\
&= \int_{Y_R}^{Y_L}\int_{-\infty}^{+\infty}\left[\frac{\partial \hat{\eta}}{\partial X} + \frac{1}{2C}\frac{\mathrm{d}C}{\mathrm{d}X}\hat{\eta}\right]\mathrm{d}\xi\mathrm{d}Y + \frac{\mathrm{d}Y_L}{\mathrm{d}X}\int_{-\infty}^{+\infty}\hat{\eta}|_{Y=Y_L}\mathrm{d}\xi - \frac{\mathrm{d}Y_R}{\mathrm{d}X}\int_{-\infty}^{+\infty}\hat{\eta}|_{Y=Y_R}\mathrm{d}\xi \\
&= \frac{\beta}{2}\int_{Y_R}^{Y_L}\int_{-\infty}^{+\infty}V\mathrm{d}\xi\mathrm{d}Y + \frac{1}{2}\frac{\mathrm{d}Y_L}{\mathrm{d}X}\int_{-\infty}^{+\infty}\hat{\eta}|_{Y=Y_L}\mathrm{d}\xi - \frac{1}{2}\frac{\mathrm{d}Y_R}{\mathrm{d}X}\int_{-\infty}^{+\infty}\hat{\eta}|_{Y=Y_R}\mathrm{d}\xi.
\end{aligned} \tag{4.4}$$

If the rotation is absent, i.e. $\beta = 0$, from (4.3) and (4.4), we find that

$$\mathcal{I} = (C/W)^{1/2}\int_{Y_R}^{Y_L}\int_{-\infty}^{+\infty}\hat{\eta}\,\mathrm{d}\xi\mathrm{d}Y \tag{4.5}$$

($W(X) = Y_L(X) - Y_R(X)$ is the width of the channel) is the first-order invariant (in amplitude). Unfortunately, when $\beta \neq 0$, we fail to find the corresponding first-order invariant from the uKP equation (the first-order invariant may not exist in this case). However, we do find the second-order invariant defined as

$$\mathscr{J} = C \int_{Y_R}^{Y_L} \int_{-\infty}^{+\infty} \hat{\eta}^2 \mathrm{d}\xi \mathrm{d}Y \tag{4.6}$$

for all β by multiplying (4.1a) by $\hat{\eta}$ and integrating the resulting equation over $[-\infty < \xi < +\infty; Y_R \leqslant Y \leqslant Y_L]$.

In the physical coordinates, the dimensionless mass $\mathscr{M}$ and energy $\mathscr{E}$ are defined as

$$\mathscr{M} = \epsilon^{1/2} \int_{-\infty}^{+\infty} \int_{y_r}^{y_l} \hat{\eta} \mathrm{d}x\mathrm{d}y, \qquad \mathscr{E} = \epsilon^{1/2} \int_{-\infty}^{+\infty} \int_{y_r}^{y_l} \hat{\eta}^2 \mathrm{d}x\mathrm{d}y. \tag{4.7a,b}$$

From transformation (2.9a), we have $\mathrm{d}x = C\mathrm{d}\xi, \mathrm{d}y = \epsilon^{-1/2}\mathrm{d}Y$. Thus, in terms of the moving coordinates, $\mathscr{M}$ and $\mathscr{E}$ become

$$\mathscr{M} = C \int_{Y_R}^{Y_L} \int_{-\infty}^{+\infty} \hat{\eta} \mathrm{d}\xi \mathrm{d}Y, \qquad \mathscr{E} = C \int_{Y_R}^{Y_L} \int_{-\infty}^{+\infty} \hat{\eta}^2 \mathrm{d}\xi \mathrm{d}Y. \tag{4.8a,b}$$

Therefore, the second-order invariant $\mathscr{J}$ measures the energy, whereas the first-order invariant $\mathscr{I}$ (when the rotation is absent) measures the mass only if CW is a constant. It follows that a locally confined solution of the uKP equation (2.14) with the boundary conditions (2.16) will conserve the energy, but in general will not conserve the mass, even when the first-order invariant exists. This is a direct consequence of the neglect of the weak backward propagating wave field excited by the variation of the channel (and perhaps by the rotation if it exists) in the uKP equation. The energy of the neglected backward propagating waves is of a higher order, whereas the mass of the backward propagating waves is of the first order and has a cumulative effect. To ensure that both $\mathscr{M}$ and $\mathscr{I}$ (when $\beta = 0$) are conserved, the neglected weak backward propagating wave field has to be taken into account (for the KdV equation case, see Miles 1979 and Knickerbocker & Newell 1985).

If the initial condition is given by

$$\hat{\eta}(\xi, 0, Y) = \frac{2D_1}{C^2 D_{-2}} k^2 \mathrm{sech}^2 \left\{\tfrac{1}{2}(6/\alpha)^{1/2} \left[k\xi + l(6D_1)^{1/2} Y/C\right]\right\} \exp\left(-\beta Y/C\right), \tag{4.9a}$$

which represents a normal ($l = 0$) or an oblique ($l \neq 0$) incident solitary wave and satisfies (4.3) at $X = 0$, and the solution evolving from this initial condition is assumed to be locally confined, we can evaluate the invariants $\mathscr{I}$ (when $\beta = 0$) and $\mathscr{J}$ analytically:

$$\mathscr{I} = (C/W)^{1/2} \int_{Y_R}^{Y_L} \int_{-\infty}^{+\infty} \hat{\eta}(\xi, 0, Y) \mathrm{d}\xi \mathrm{d}Y = \frac{8kD_1}{C^2 D_{-2}} (CW\alpha/6)^{1/2} \qquad (\beta = 0) \tag{4.9b}$$

and

$$\begin{aligned} \mathscr{J} &= C \int_{Y_R}^{Y_L} \int_{-\infty}^{+\infty} \hat{\eta}^2(\xi, 0, Y) \mathrm{d}\xi \mathrm{d}Y \\ &= (\alpha/6)^{1/2} \frac{16k^3 D_1^2}{3\beta C^2 D_{-2}^2} \left[\exp(-2\beta Y_R/C) - \exp(-2\beta Y_L/C)\right], \end{aligned} \tag{4.9c}$$

where all the functions in (4.9) are evaluated at $X = 0$.

5. Summary and concluding remarks

Using the reductive perturbation method, we have derived the unified KP (uKP) equation for surface and interfacial waves propagating in a rotating channel with varying topography and sidewalls. The effect of a steady background current field on wave propagation has also been taken into account. The uKP equation includes most of the existing KP-type equations for surface water waves and interfacial waves as special cases. The Painlevé PDE test has been used to search for the conditions for the uKP equation to be completely integrable. When these conditions are satisfied, transformations have been found to reduce the uKP equation to one of known integrable equations: the KP, the KdV or the cKdV equations. As a result, for certain topographies and sidewalls, analytical solutions for solitary-wave propagation can be obtained in the absence of rotation. The integral invariants associated with the uKP equation for waves propagating in a varying channel have been obtained and their relations with mass conservation and energy conservation have been discussed.

When the uKP equation is completely integrable, there are several powerful analytical techniques for obtaining many classes of solutions (soliton, multisoliton, periodic solutions, etc.) to the uKP equation, such as inverse scattering transform, Bäcklund transformations, Hirota's method and symmetry reduction. Unfortunately, the conditions for the uKP equation to be completely integrable are very restrictive. They require that no rotation exist, the variation of the topography should be weak ($h^- =$ constant) and the topography (and the current field if it exists) in the transversal direction be a linear function of Y only. In addition, the sidewalls usually will interfere with the integrability. Therefore, to apply the uKP equation to more complex situations, one needs to solve the equation numerically. An efficient and accurate numerical scheme has been developed to solve the uKP equation by using the Petrov–Galerkin finite element method and some numerical results have been obtained. The numerical study of the uKP equation will be reported in the near future.

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Appendix. Detailed derivation of the uKP equation

In this Appendix we present the detailed perturbation procedures and analyses which lead to the uKP equation. Substituting the perturbation expansions, (2.12), into the governing equations (2.11), we obtain a sequence of initial-boundary-value problems.

A.1. *The zeroth-order problem*

The zeroth-order problem is

$$\frac{1}{C}\frac{\partial u_0^\pm}{\partial \xi} + \frac{\partial w_0^\pm}{\partial Z} = 0, \tag{A 1a}$$

$$\frac{\partial u_0^\pm}{\partial \xi} - \frac{1}{C}\frac{\partial p_0^\pm}{\partial \xi} = 0, \tag{A 1b}$$

$$\frac{\partial v_0^\pm}{\partial \xi} - \frac{\partial p_0^\pm}{\partial Y} - \beta u_0^\pm = 0, \tag{A 1c}$$

$$\frac{\partial p_0^\pm}{\partial Z} = 0, \tag{A 1d}$$

$$w_0^\pm + \frac{\partial \eta_0}{\partial \xi} = 0 \quad \text{on} \quad Z = 0, \tag{A 1e}$$

$$\rho^+ p_0^+ - \rho^- p_0^- + \eta_0 = 0 \quad \text{on} \quad Z = 0, \tag{A 1f}$$

$$w_0^+ = 0 \quad \text{on} \quad Z = h^+, \tag{A 1g}$$

$$w_0^- = 0 \quad \text{on} \quad Z = -h^-, \tag{A 1h}$$

$$v_0^\pm = u_0^\pm \frac{\mathrm{d}Y}{\mathrm{d}X} \quad \text{on} \quad Y = Y_R(X),\ Y_L(X). \tag{A 1i}$$

From (A 1*a*)–(A 1*h*), we obtain the following solution forms:

$$p_0^\pm(\xi, X, Y) = \mp C^2 \eta_0 / h^\pm + \mathscr{F}_p^\pm(X, Y), \tag{A 2a}$$

$$u_0^\pm(\xi, X, Y, Z) = \mp C \eta_0 / h^\pm + \mathscr{F}_u^\pm(X, Y, Z), \tag{A 2b}$$

$$\frac{\partial v_0^\pm(\xi, X, Y, Z)}{\partial \xi} = \mp \frac{C^2}{h^\pm}\left[\frac{\partial \eta_0}{\partial Y} + \frac{\beta}{C}\eta_0\right] + \frac{\partial \mathscr{F}_p^\pm}{\partial Y} + \beta \mathscr{F}_u^\pm, \tag{A 2c}$$

$$w_0^\pm(\xi, X, Y, Z) = \frac{(-h^\pm \pm Z)}{h^\pm}\frac{\partial \eta_0}{\partial \xi}, \tag{A 2d}$$

$$\rho^+ \mathscr{F}_p^+ = \rho^- \mathscr{F}_p^-, \tag{A 2e}$$

where $\mathscr{F}_p^\pm$ and $\mathscr{F}_u^\pm$ are constants of integration with respect to ξ. In view of transformation (2.9*a*), these functions are independent of the physical time t. We could also integrate (A 2*c*) with respect to ξ and introduce another set of constants of integration, $\mathscr{F}_v^\pm(X, Y, Z)$. However, to obtain the evolution equation for η_0, $\partial_\xi v_0^\pm$ will be used directly. These constants of integration, $\mathscr{F}_u^\pm$, $\mathscr{F}_v^\pm$ and $\mathscr{F}_p^\pm$, represent a background steady current field (with non-vanishing vorticity in general), which is of the same order of magnitude as the velocity field associated with the wave motion. Note that we have already expressed all zeroth-order solutions in terms of η_0 which can be determined from the solvability condition of the first-order problem.

A.2. *The first-order problem*

The governing equations and boundary conditions for the first-order problem can be written as

$$\frac{1}{C}\frac{\partial u_1^\pm}{\partial \xi} + \frac{\partial w_1^\pm}{\partial Z} = -\frac{\partial u_0^\pm}{\partial X} - \frac{\partial v_0^\pm}{\partial Y}, \tag{A 3a}$$

$$\frac{\partial u_1^\pm}{\partial \xi} - \frac{1}{C}\frac{\partial p_1^\pm}{\partial \xi} = \frac{u_0^\pm}{C}\frac{\partial u_0^\pm}{\partial \xi} + w_0^\pm \frac{\partial u_0^\pm}{\partial Z} + \frac{\partial p_0^\pm}{\partial X} - \beta v_0^\pm, \tag{A 3b}$$

$$\frac{\partial v_1^\pm}{\partial \xi} - \frac{\partial p_1^\pm}{\partial Y} - \beta u_1^\pm = \frac{u_0^\pm}{C}\frac{\partial v_0^\pm}{\partial \xi} + w_0^\pm \frac{\partial v_0^\pm}{\partial Z}, \tag{A 3c}$$

$$\frac{\partial p_1^\pm}{\partial Z} = \alpha \frac{\partial w_0^\pm}{\partial \xi}, \tag{A 3d}$$

$$w_1^\pm + \frac{\partial \eta_1}{\partial \xi} = \frac{u_0^\pm}{C}\frac{\partial \eta_0}{\partial \xi} - \frac{\partial w_0^\pm}{\partial Z}\eta_0 \quad \text{on} \quad Z = 0, \tag{A 3e}$$

$$\rho^+ p_1^+ - \rho^- p_1^- + \eta_1 = 0 \quad \text{on} \quad Z = 0, \tag{A 3f}$$

$$w_1^+ = 0 \quad \text{on} \quad Z = h^+, \tag{A 3g}$$

$$w_1^- = \frac{\partial w_0^-}{\partial Z} B - u_0^- \frac{\mathrm{d}h^-}{\mathrm{d}X} \quad \text{on} \quad Z = -h^-, \tag{A 3h}$$

$$v_1^\pm = u_1^\pm \frac{\mathrm{d}Y}{\mathrm{d}X} \quad \text{on} \quad Y = Y_R(X),\ Y_L(X), \tag{A 3i}$$

in which (A 1*d*) has been used to obtain (A 3*f*). Substituting (A 2*d*) into (A 3*d*) and integrating the resulting equations from 0 to Z, we obtain the vertical profiles of the pressure $p_1^\pm$,

$$p_1^\pm = p_{10}^\pm \pm \frac{\alpha}{h^\pm}\left(\frac{Z^2}{2} \mp h^\pm Z\right)\frac{\partial^2 \eta_0}{\partial \xi^2}, \tag{A 4}$$

where $p_{10}^\pm(\xi, X, Y) = p_1^\pm(\xi, X, Y, Z)|_{Z=0}$. All the other first-order quantities will be expressed in terms of η_0 and $p_{10}^\pm$.

Substituting (A 2) and (A 4) into (A 3*b*), we obtain

$$\frac{\partial u_1^\pm}{\partial \xi} = \frac{1}{C}\frac{\partial p_{10}^\pm}{\partial \xi} \pm \frac{\alpha}{C h^\pm}\left(\frac{Z^2}{2} \mp h^\pm Z\right)\frac{\partial^3 \eta_0}{\partial \xi^3} + \frac{C}{2(h^\pm)^2}\frac{\partial \eta_0^2}{\partial \xi} \mp \frac{\partial}{\partial X}\left(\frac{C^2}{h^\pm}\eta_0\right)$$
$$- \beta v_0^\pm \mp \frac{\mathscr{F}_u^\pm}{h^\pm}\frac{\partial \eta_0}{\partial \xi} + \frac{(-h^\pm \pm Z)}{h^\pm}\frac{\partial \eta_0}{\partial \xi}\frac{\partial \mathscr{F}_u^\pm}{\partial Z} + \frac{\partial \mathscr{F}_p^\pm}{\partial X}. \tag{A 5}$$

Replacing $\partial_\xi u_1^\pm$ in the continuity equations (A 3*a*) by (A 5), integrating the resulting equations with respect to Z from 0 to h^+ for w_1^+ and from $-h^-$ to 0 for w_1^-, and applying the boundary conditions (A 3*g*) and (A 3*h*), we obtain both upper- and lower-layer vertical velocity components evaluated along $Z = 0$:

$$w_{10}^+ = \frac{h^+}{C^2}\frac{\partial p_{10}^+}{\partial \xi} - \frac{\alpha(h^+)^2}{3C^2}\frac{\partial^3 \eta_0}{\partial \xi^3} + \frac{1}{2h^+}\frac{\partial \eta_0^2}{\partial \xi} - 2C\frac{\partial \eta_0}{\partial X} - 3\frac{\mathrm{d}C}{\mathrm{d}X}\eta_0 + \frac{\partial \mathscr{U}^+}{\partial X}$$
$$- \int_0^{h^+}\left[\frac{\beta v_0^+}{C} - \frac{\partial v_0^+}{\partial Y}\right]\mathrm{d}Z - \frac{2\mathscr{U}^+}{Ch^+}\frac{\partial \eta_0}{\partial \xi} + \frac{\mathscr{F}_u^+|_{Z=0}}{C}\frac{\partial \eta_0}{\partial \xi} + \frac{h^+}{C}\frac{\partial \mathscr{F}_p^+}{\partial X}, \tag{A 6a}$$

$$w_{10}^- = -\frac{h^-}{C^2}\frac{\partial p_{10}^-}{\partial \xi} - \frac{\alpha(h^-)^2}{3C^2}\frac{\partial^3 \eta_0}{\partial \xi^3} - \frac{1}{2h^-}\frac{\partial \eta_0^2}{\partial \xi} - 2C\frac{\partial \eta_0}{\partial X}$$
$$- \left[3\frac{\mathrm{d}C}{\mathrm{d}X} - \frac{C}{h^-}\frac{\mathrm{d}h^-}{\mathrm{d}X}\right]\eta_0 + \int_{-h^-}^{0}\left[\frac{\beta v_0^-}{C} - \frac{\partial v_0^-}{\partial Y}\right]\mathrm{d}Z$$
$$- \frac{2\mathscr{U}^-}{Ch^-}\frac{\partial \eta_0}{\partial \xi} + \frac{\mathscr{F}_u^-|_{Z=0}}{C}\frac{\partial \eta_0}{\partial \xi} - \frac{h^-}{C}\frac{\partial \mathscr{F}_p^-}{\partial X} - \frac{\partial \mathscr{U}^-}{\partial X} - \frac{B}{h^-}\frac{\partial \eta_0}{\partial \xi}, \tag{A 6b}$$

where $w_{10}^\pm(\xi, X, Y) = w_1^\pm(\xi, X, Y, Z)|_{Z=0}$ and

$$\mathscr{U}^+(X, Y) = \int_0^{h^+} \mathscr{F}_u^+ \mathrm{d}Z, \qquad \mathscr{U}^-(X, Y) = \int_{-h^-}^{0} \mathscr{F}_u^- \mathrm{d}Z, \tag{A 7a,b}$$

which are the volume fluxes of the background current field in the x-direction in the upper- and lower-layer, respectively. Differentiating (A 6) with respect to ξ and substituting (A 2*c*) into the resulting equations, we find

$$\frac{\partial w_{10}^{+}}{\partial \xi}=\frac{h^{+}}{C^{2}}\frac{\partial^{2} p_{10}^{+}}{\partial \xi^{2}}-\frac{\alpha(h^{+})^{2}}{3C^{2}}\frac{\partial^{4}\eta_{0}}{\partial \xi^{4}}+\frac{1}{2h^{+}}\frac{\partial^{2}\eta_{0}^{2}}{\partial \xi^{2}}-2C\frac{\partial^{2}\eta_{0}}{\partial X\partial \xi}-3\frac{\mathrm{d}C}{\mathrm{d}X}\frac{\partial \eta_{0}}{\partial \xi}$$
$$-C^{2}\frac{\partial^{2}\eta_{0}}{\partial Y^{2}}+\beta^{2}\eta_{0}-\frac{\beta}{C}\left[h^{+}\frac{\partial \mathscr{F}_{p}^{+}}{\partial Y}+\beta\mathscr{U}^{+}\right]+h^{+}\frac{\partial^{2}\mathscr{F}_{p}^{+}}{\partial Y^{2}}$$
$$+\beta\frac{\partial\mathscr{U}^{+}}{\partial Y}-\frac{2\mathscr{U}^{+}}{Ch^{+}}\frac{\partial^{2}\eta_{0}}{\partial\xi^{2}}+\frac{\mathscr{F}_{u}^{+}|_{Z=0}}{C}\frac{\partial^{2}\eta_{0}}{\partial \xi^{2}}, \qquad (\mathrm{A}\,8a)$$

$$\frac{\partial w_{10}^{-}}{\partial \xi}=-\frac{h^{-}}{C^{2}}\frac{\partial^{2} p_{10}^{-}}{\partial \xi^{2}}-\frac{\alpha(h^{-})^{2}}{3C^{2}}\frac{\partial^{4}\eta_{0}}{\partial \xi^{4}}-\frac{1}{2h^{-}}\frac{\partial^{2}\eta_{0}^{2}}{\partial \xi^{2}}-2C\frac{\partial^{2}\eta_{0}}{\partial X\partial \xi}$$
$$-\left[3\frac{\mathrm{d}C}{\mathrm{d}X}-\frac{C}{h^{-}}\frac{\mathrm{d}h^{-}}{\mathrm{d}X}\right]\frac{\partial\eta_{0}}{\partial\xi}-C^{2}\frac{\partial^{2}\eta_{0}}{\partial Y^{2}}+\beta^{2}\eta_{0}+\frac{\beta}{C}\left[h^{-}\frac{\partial \mathscr{F}_{p}^{-}}{\partial Y}+\beta\mathscr{U}^{-}\right]$$
$$-h^{-}\frac{\partial^{2}\mathscr{F}_{p}^{-}}{\partial Y^{2}}-\beta\frac{\partial\mathscr{U}^{-}}{\partial Y}-\frac{2\mathscr{U}^{-}}{Ch^{-}}\frac{\partial^{2}\eta_{0}}{\partial\xi^{2}}+\frac{\mathscr{F}_{u}^{-}|_{Z=0}}{C}\frac{\partial^{2}\eta_{0}}{\partial \xi^{2}}-\frac{B}{h^{-}}\frac{\partial^{2}\eta_{0}}{\partial \xi^{2}}. \qquad (\mathrm{A}\,8b)$$

On the other hand, from the dynamic interfacial condition (A 3f), we have

$$\eta_{1}=\rho^{-}p_{10}^{-}-\rho^{+}p_{10}^{+}. \qquad (\mathrm{A}\,9)$$

Substituting this expression and the zeroth-order solutions (A 2) into the kinematic interfacial conditions (A 3e), we have

$$w_{10}^{\pm}+\rho^{-}\frac{\partial p_{10}^{-}}{\partial\xi}-\rho^{+}\frac{\partial p_{10}^{+}}{\partial\xi}=\mp\frac{1}{h^{\pm}}\frac{\partial\eta_{0}^{2}}{\partial\xi}+\frac{\mathscr{F}_{u}^{\pm}|_{Z=0}}{C}\frac{\partial\eta_{0}}{\partial\xi}. \qquad (\mathrm{A}\,10)$$

Differentiating equations (A 10) with respect to ξ and replacing $\partial_{\xi}w_{10}^{\pm}$ by (A 8), we obtain

$$\rho^{-}\left[\frac{\partial^{2}p_{10}^{-}}{\partial\xi^{2}}+\frac{h^{+}}{h^{-}}\frac{\partial^{2}p_{10}^{+}}{\partial\xi^{2}}\right]=\frac{\alpha(h^{+})^{2}}{3C^{2}}\frac{\partial^{4}\eta_{0}}{\partial\xi^{4}}-\frac{3}{2h^{+}}\frac{\partial^{2}\eta_{0}^{2}}{\partial\xi^{2}}+2C\frac{\partial^{2}\eta_{0}}{\partial X\partial\xi}+3\frac{\mathrm{d}C}{\mathrm{d}X}\frac{\partial\eta_{0}}{\partial\xi}-\beta^{2}\eta_{0}$$
$$+C^{2}\frac{\partial^{2}\eta_{0}}{\partial Y^{2}}+\frac{\beta}{C}\left[h^{+}\frac{\partial\mathscr{F}_{p}^{+}}{\partial Y}+\beta\mathscr{U}^{+}\right]-h^{+}\frac{\partial^{2}\mathscr{F}_{p}^{+}}{\partial Y^{2}}-\beta\frac{\partial\mathscr{U}^{+}}{\partial Y}+\frac{2\mathscr{U}^{+}}{Ch^{+}}\frac{\partial^{2}\eta_{0}}{\partial\xi^{2}}, \qquad (\mathrm{A}\,11a)$$

$$-\rho^{+}\left[\frac{h^{-}}{h^{+}}\frac{\partial^{2}p_{10}^{-}}{\partial\xi^{2}}+\frac{\partial^{2}p_{10}^{+}}{\partial\xi^{2}}\right]=\frac{\alpha(h^{-})^{2}}{3C^{2}}\frac{\partial^{4}\eta_{0}}{\partial\xi^{4}}+\frac{3}{2h^{-}}\frac{\partial^{2}\eta_{0}^{2}}{\partial\xi^{2}}+2C\frac{\partial^{2}\eta_{0}}{\partial X\partial\xi}$$
$$+\left[3\frac{\mathrm{d}C}{\mathrm{d}X}-\frac{C}{h^{-}}\frac{\mathrm{d}h^{-}}{\mathrm{d}X}\right]\frac{\partial\eta_{0}}{\partial\xi}-\beta^{2}\eta_{0}+C^{2}\frac{\partial^{2}\eta_{0}}{\partial Y^{2}}-\frac{\beta}{C}\left[h^{-}\frac{\partial\mathscr{F}_{p}^{-}}{\partial Y}+\beta\mathscr{U}^{-}\right]$$
$$+h^{-}\frac{\partial^{2}\mathscr{F}_{p}^{-}}{\partial Y^{2}}+\beta\frac{\partial\mathscr{U}^{-}}{\partial Y}+\frac{2\mathscr{U}^{-}}{Ch^{-}}\frac{\partial^{2}\eta_{0}}{\partial\xi^{2}}+\frac{B}{h^{-}}\frac{\partial^{2}\eta_{0}}{\partial\xi^{2}}. \qquad (\mathrm{A}\,11b)$$

Multiplying (A 11a) by $\rho^{+}h^{-}/\rho^{-}h^{+}$ and adding the resulting equation to (A 11b), all terms on the left-hand side are cancelled and all terms on the right-hand side yield

$$C^{1/2}\frac{\partial}{\partial X}\left(C^{1/2}\frac{\partial\eta_{0}}{\partial\xi}\right)+\frac{3C^{2}}{4}D_{-2}\frac{\partial^{2}\eta_{0}^{2}}{\partial\xi^{2}}+\frac{\alpha D_{1}}{6}\frac{\partial^{4}\eta_{0}}{\partial\xi^{4}}+\frac{C^{2}}{2}\frac{\partial^{2}\eta_{0}}{\partial Y^{2}}-\frac{\beta^{2}}{2}\eta_{0}$$
$$+\left[\frac{\rho^{-}BC^{2}}{2(h^{-})^{2}}+C\mathscr{N}_{2}\right]\frac{\partial^{2}\eta_{0}}{\partial\xi^{2}}=\frac{\beta^{2}C}{2}\mathscr{N}_{1}-\frac{\beta C^{2}}{2}\frac{\partial\mathscr{N}_{1}}{\partial Y}, \qquad (\mathrm{A}\,12)$$

where

$$D_{n}(X)=\rho^{-}(h^{-})^{n}+(-1)^{n-1}\rho^{+}(h^{+})^{n},\quad (n=1,\ -2), \qquad (\mathrm{A}\,13a)$$

$$\mathcal{N}_n(X,Y) = \rho^- \mathcal{U}^-/(h^-)^n + (-1)^n \rho^+ \mathcal{U}^+/(h^+)^n, \quad (n = 1, -2), \tag{A 13b}$$

and (A 2*e*) has been used.

The boundary conditions for η_0 on the sidewalls are obtained by differentiating (A 1*i*) with respect to ξ and substituting (A 2*b*) and (A 2*c*) into the resulting equations:

$$\frac{\partial \eta_0}{\partial Y} + \frac{\beta}{C}\eta_0 - \frac{1}{C}\frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\partial \eta_0}{\partial \xi} = \beta\left(\rho^+ \mathcal{F}_u^+ - \rho^- \mathcal{F}_u^-\right) \quad \text{on} \quad Y = Y_R(X),\ Y_L(X), \tag{A 14a}$$

with

$$\left[h^+ + \frac{\rho^+ h^-}{\rho^-}\right]\frac{\partial \mathcal{F}_p^+}{\partial Y} + \beta\left(h^+ \mathcal{F}_u^+ + h^- \mathcal{F}_u^-\right) = 0 \quad \text{on} \quad Y = Y_R(X),\ Y_L(X). \tag{A 14b}$$

Both (A 14*a*) and (A 14*b*) require that $\mathcal{F}_u^{\pm}$ be independent of Z on $Y = Y_R(X)$ and $Y = Y_L(X)$. Thus, from (A 7), $\mathcal{F}_u^{\pm} = \mathcal{U}^{\pm}/h^{\pm}$ on the sidewalls. Therefore, from the governing equation (A 12) and the boundary conditions (A 14), the influence of the steady current field on the wave field is through the fluxes in the x-direction in both layers $\mathcal{U}^{\pm}$ (note that the counterparts in the y-direction are $O(\epsilon^{1/2})$, see (2.12*b*)). If $\mathcal{U}^+ = \mathcal{U}^- = 0$, the weak current field does not have any impact on the wave field (up to $O(\epsilon)$).

A.3. *Mean interfacial displacement and the uKP equation*

When the rotation is present and the averaged mass fluxes of the steady current in the x-direction are not identical in the upper and lower layers, i.e. $\beta \neq 0$ and $\mathcal{N}_1 \neq 0$ (which means $\int_{-h^-}^{0} \rho^- \mathcal{F}_u^- \mathrm{d}Z/h^- \neq \int_0^{h^+} \rho^+ \mathcal{F}_u^+ \mathrm{d}Z/h^+$), equation (A 12) is an inhomogeneous differential equation for the interfacial displacement η_0, whereas (A 14*a*) are inhomogeneous boundary conditions. Since the terms on the right-hand side of (A 12) are independent of ξ, a solution to (A 12) can be written as

$$\eta_0(\xi, X, Y) = \hat{\eta}(\xi, X, Y) + \bar{\eta}(X, Y), \tag{A 15}$$

where the steady part (independent of the physical time t)

$$\bar{\eta}(X,Y) = -\beta \exp\left(-\beta Y/C\right)\int_{Y_R}^{Y} \mathcal{N}_1(X,Y')\exp\left(\beta Y'/C\right)\mathrm{d}Y' \tag{A 16}$$

is the particular solution to (A 12) which also satisfies the boundary conditions (A 14*a*), whereas the unsteady part $\hat{\eta}$ is the solution to the homogeneous equation

$$C^{1/2}\frac{\partial}{\partial X}\left(C^{1/2}\frac{\partial \hat{\eta}}{\partial \xi}\right) + \frac{3C^2}{4}D_{-2}\frac{\partial^2 \hat{\eta}^2}{\partial \xi^2} + \frac{\alpha D_1}{6}\frac{\partial^4 \hat{\eta}}{\partial \xi^4} + \frac{C^2}{2}\frac{\partial^2 \hat{\eta}}{\partial Y^2} - \frac{\beta^2}{2}\hat{\eta} + \left[\frac{\rho^- B C^2}{2(h^-)^2} + C\mathcal{N}_2 + \frac{3C^2}{2}D_{-2}\bar{\eta}\right]\frac{\partial^2 \hat{\eta}}{\partial \xi^2} = 0. \tag{A 17}$$

The boundary conditions for $\hat{\eta}$ are also homogeneous:

$$\frac{\partial \hat{\eta}}{\partial Y} + \frac{\beta}{C}\hat{\eta} - \frac{1}{C}\frac{\mathrm{d}Y}{\mathrm{d}X}\frac{\partial \hat{\eta}}{\partial \xi} = 0 \quad \text{on} \quad Y = Y_R(X),\ Y_L(X). \tag{A 18}$$

Thus, when the averaged mass fluxes of the steady current in the x-direction are not identical in the upper and lower layers, rotation will cause a mean interfacial displacement, $Z = \bar{\eta}(X,Y)$. Equations (A 17) and (A 18) are the uKP equation and the corresponding boundary conditions on the sidewalls of a channel.

REFERENCES

ABLOWITZ, M. J. & CLARKSON, P. A. 1991 *Solitons, Nonlinear Evolution Equations and Inverse Scattering*. Cambridge University Press.

AKYLAS, T. R. 1994 Three-dimensional long water-wave phenomena. *Ann. Rev. Fluid Mech.* **26**, 191–210.

BRUGARINO, T. & GRECO, A. M. 1991 Painlevé analysis and reducibility to the canonical form for the generalized Kadomtsev–Petviashvili equation. *J. Math. Phys.* **32**, 69–71.

CALOGERO, F. & DEGASPERIS, A. 1978 Solution by the spectral-transform method of a nonlinear evolution equation including as a special case the cylindrical KdV equation. *Lett. Nuovo Cim.* **19**, 150–154.

CHEN, Y. & WEN, S. 1987 Traveling wave solutions to the two-dimensional Korteweg–de Vries equation. *J. Maths Anal. Appl.* **127**, 226–236.

CLARKSON, P. A. 1990 Painlevé analysis and the complete integrability of a generalized variable-coefficient Kadomtsev–Petviashvili equation. *IMA J. Appl. Maths* **44**, 27–53.

DAVID, D., LEVI, D. & WINTERNITZ, P. 1987 Integrable nonlinear equations for water waves in straits of varying depth and width. *Stud. Appl. Maths* **76**, 133–168.

DAVID, D., LEVI, D. & WINTERNITZ, P. 1989 Solitons in shallow seas of variable depth and in marine straits. *Stud. Appl. Maths* **80**, 1–23.

DJORDJEVIC, V. D. & REDEKOPP, L. G. 1978 The fission and disintegration of internal solitary waves moving over two-dimensional topography. *J. Phys. Oceanogr.* **8**, 1016–1024.

DRAZIN, P. G. & JOHNSON, R. S. 1989 *Solitons: An Introduction.* Cambridge University Press.

FREEMAN, N. C. 1980 Soliton interactions in two dimensions. *Adv. Appl. Mech.* **20**, 1–37.

GARDNER, C. S., GREENE, J. M., KRUSKAL, M. D. & MIURA, R. M. 1967 Method for solving the KdV equation. *Phys. Rev. Lett.* **19**, 1095–1097.

GRIMSHAW, R. 1985 Evolution equations for weakly nonlinear, long internal waves in a rotating fluid. *Stud. Appl. Maths* **73**, 1–33.

GRIMSHAW, R. & MELVILLE, W. K. 1989 On the Derivation of the modified Kadomtsev–Petviashvili equation. *Stud. Appl. Maths* **80**,183–202.

GRIMSHAW, R. & TANG, S. 1990 The rotation-modified Kadomtsev–Petviashvili equation: An analytical and numerical study. *Stud. Appl. Maths* **83**, 223–248.

HAMMACK, J., SCHEFFNER, N. & SEGUR, H. 1989 Two-dimensional periodic waves in shallow water. *J. Fluid Mech.* **209**, 567–589.

IIZUKA, T. & WADATI, M. 1992 Shallow water waves over an uneven bottom and an inhomogeneous KP equation. *Chaos, Solitons and Fractals* **2**, 575–582.

JOHNSON, R. S. 1973 On the development of a solitary wave moving over an uneven bottom. *Proc. Camb. Phil. Soc.* **73**, 183–203.

JOHNSON, R. S. 1980 Water waves and Korteweg–de Vries equations. *J. Fluid Mech.* **97**, 701–719.

KADOMTSEV, B. B. & PETVIASHVILI, V. I. 1970 On the stability of solitary waves in weakly dispersing media. *Sov. Phys. Dokl.* **15**, 539–541.

KAKUTANI, T. 1971 Effect of an uneven bottom on gravity waves. *J. Phys. Soc. Japan* **30**, 272–276.

KATSIS, C. & AKYLAS, T. R. 1987 Solitary internal waves in a rotating channel: A numerical study. *Phys. Fluids* **30**, 297–301.

KNICKERBOCKER, C. J. & NEWELL, A. C.1985 Reflections from solitary waves in channels of decrease depth. *J. Fluid Mech.* **153**, 1–16.

MATHEW, J. & AKYLAS, T. R. 1990 On three-dimensional long water waves in a channel with sloping sidewalls. *J. Fluid Mech.* **215**, 289–307.

MILES, J. W. 1979 On the Korteweg–de Vries equation for a gradually varying channel. *J. Fluid Mech.* **91**, 181–190.

WEISS, J., TABOR, M. & CARNEVALE, G. 1983 The Painlevé property for partial differential equations. *J. Math. Phys.* **24**, 522–526.